Near Optimal Risk Budgeting with Target Returns: Exact and Tight Approximation Algorithms

Viraat Singh* Ali Hirsa†

Abstract

Traditional risk parity maximizes risk diversification in a portfolio, an extremely desirable characteristic which has been explored extensively. However, algorithms for risk parity are return agnostic, i.e., the risk measure to measure risk in a portfolio is chosen to be portfolio volatility for mathematical and computational convenience. Efforts in portfolio optimization literature have been made to incorporate expected returns into risk parity, but due to the complex nature of the problem, these efforts do not provide guarantees on how much risk is diversified in a portfolio. Moreover, efforts in this direction provide relaxations to return based risk parity as an optimization problem which can show arbitrarily bad risk diversification. In this paper, we provide algorithms that find portfolios that are closest, in some mathematical sense, to a risk parity portfolio along with a return target. This enables us to overcome the return-agnostic nature of risk parity while finding portfolios with good risk diversification.

`Keywords:` risk parity; risk budgeting; portfolio selection; convex optimization for portfolio selection; convex framework for return-based risk parity; risk diversification; guarantees on risk spread

1 Problem Introduction

We denote by a risk measure $\mathcal{R}(x) : \mathbb{R}^n \to \mathbb{R}$ to be a positive homogeneous function of degree one in portfolio weights as in [1] and [2]. Examples of positively homogeneous risk measures include portfolio volatility, value-at-risk (`VaR`) and any other coherent risk measures such as conditional value-at-risk (`CVaR`) [1]. Equal Risk Contribution (`ERC`), which was first coined by [9], [8] and then extensively studied by [19] and [15]. `ERC` aims to find a portfolio x that assigns equal risk share $\mathcal{RC}_i(x)$ to all assets being traded, indexed by set $[n]$. We define our risk measure for a given covariance matrix Σ to be volatility, that is,

$$\mathcal{R}(x) = \sqrt{x^\top \Sigma x}$$

and define risk contribution of asset $i \in [n] = \{1, 2, \ldots, n\}$, $\mathcal{RC}_i(x)$ by,

$$\mathcal{RC}_i(x) = x_i \frac{\partial \mathcal{R}(x)}{\partial x} = \frac{\partial \sqrt{x^\top \Sigma x}}{\partial x_i} x_i = \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} x_i$$

Moreover, we have a Euler decomposition for a risk measure [14], [23] defined as above, that is, for a homogeneous function $f(x)$ of degree k, we have that,

$$k f(x) = \nabla f(x)^\top x$$

implying the following decomposition of our risk measure,

$$\mathcal{R}(x) = \sum_{i=1}^{n} \mathcal{RC}_i(x)$$

Therefore, we define the set of risk parity or equal risk contribution (`ERC`) portfolios (without short selling) $S_{\texttt{ERC}}$ by

$$\mathcal{S}_{\texttt{ERC}} = \left\{ x : \mathcal{RC}_i(x) = \mathcal{RC}_j(x) \quad \forall i, j \in [n], \quad x \geq 0, \quad \mathbf{1}^\top x = 1 \right\}$$

The performance of these portfolios [19], [24] and their volatility [6] has been studied extensively and provide a great balance between diversification in allocation and risk ([19], [18],[22]). Significant work has been put into relaxing

*Columbia University, Industrial Engineering and Operations Research
†Columbia University, Industrial Engineering and Operations Research

the risk parity condition and budgeting the risks from each asset in portfolios called risk budgeting (`RB`) portfolios [4], [14], [13], [3].Although risk diversification is a desirable property, if we choose our risk measure as portfolio volatility defined above, our allocation turns out to be `return-agnostic`. Work has been done in designing (`RB`) portfolios with Markowitz risk measure ([12]) that takes expected returns into account ([14], [21]), however the problem is either solvable only under certain conditions or non-convex and hard to solve numerically, unlike risk parity portfolios [10]. In this paper, we present an alternative approach wherein we diversify risk with a return target as hinted in the heuristic approach of [11] and formalize it with theoretical guarantees. We adopt the principle of diversifying risk contributions to improve returns ([7],[16]) by satisfying approximate risk parity whilst providing bounds on how much the risk is diversified and taking returns into account. To this end, consider the following hypothetical example. We have two assets: one risk-free asset with an expected return of r_f and variance σ_1^2, and one risky asset with an expected return of r_R and variance σ_2^2. For simplicity, assume that the two assets are uncorrelated. The covariance matrix is thus given by

$$\Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

where we assume $\sigma_1 \ll \sigma_2$ and that $r_R \geq 2r_f$. An `ERC` portfolio yields

$$\mathcal{RC}_1(x) = x_1^2\sigma_1^2 = x_2^2\sigma_2^2 = \mathcal{RC}_2(x)$$

which gives the allocations

$$x_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2}, \quad x_2 = \frac{\sigma_1}{\sigma_1 + \sigma_2}$$

Clearly, we have $x_1 \gg x_2$. However, if a manager requires an expected return of at least $2r_f$, this would not be achievable through our portfolio. To overcome this effect, we aim to design portfolios that have a return target and are risk diversified (i.e. total risk distributed among $\mathcal{RC}_i(x)$ as evenly as possible) as per certain `spread metrics`, Θ. These spread metrics measure the risk diversification in our algorithms and are designed such that

- Can be optimized to diversify risks, we provide details of their analytical forms in the next sections
- Give risk parity portfolio x_p with return R_p for return targets less than R_p

Formalizing this notion, we have the following definition of a spread metric $\Theta : \Delta \to \mathbb{R}$, that we use to measure diversification in our portfolio,

Definition (Risk Spread Measure): *Let $r_c : \Delta \to \mathbb{R}^n$ be a function defined as,*

$$r_c(x) = (\mathcal{RC}_1(x), \ldots, \mathcal{RC}_n(x))$$

A risk spread measure Θ is a composition, $\theta \circ r_c : \Delta \to \mathbb{R}$ that is a continuous function on the domain:

$$\mathcal{X} = \left\{ x \middle| \quad \mathcal{RC}_j(x) > 0, \quad \forall j \right\}$$

s.t. risk parity portfolio is a unique solution to the problem.

$$\min_{x \in \mathcal{P}^{R_p}} \quad \Theta(x)$$

Here, $\theta(x)$ is our diversification function

Our domain $\mathcal{X}$ helps in the analysis of the algorithms that we describe in the later sections. Some examples of θ that construct a valid spread metric are $\max_j \frac{x_i}{\sum_{i=1}^n x_i}$, $\max_{i,j} \frac{x_i}{x_j}$, $\max_{i,j} \left| x_i - x_j \right|$, $\sum_{i,j}(x_i - x_j)^p$, for even p. We call θ our **diversification function**. These functions help us formalize the notion of diversification, as all of the above functions become larger when risks are 'unevenly' distributed. In other words, $\mathrm{argmin}_x\, \theta(x)$ is given by the set, $\mathcal{O} = \left\{ x \in \mathbb{R}^n \middle| \quad x_i = x_j, \quad \forall i,j \right\}$ They give us a framework under which we can evenly distribute risk and think about risk diversification. For instance, when $\theta(x) = \max_j \frac{x_i}{\sum_{i=1}^n x_i}$, we demand the maximum risk in our portfolio to be minimized w.r.t the total risk, causing the maximum risk to lie as close as possible to the average risk in the portfolio, which can be one way to define diversification. When $\Theta = \theta \circ r_c$ is minimized subject to $x \in \mathcal{P}^{R_p}$, it is easily seen that,

$$\Theta(x) \geq \frac{1}{n}, \quad \text{with equality} \iff x = x_p$$

Thus justifying the fact that Θ is a valid spread metric. Defining as in [19], $\theta(x) = \sum_{i,j}(x_i - x_j)^2$, we get a spread measure Θ that gives risk parity portfolio when minimized subject to $x \in \mathcal{P}^{R_P}$. Moreover if we demand return greater than R_p, this spread metric helps us find a risk diversified portfolio that does so. The fact that it is a spread measure follows simply from the uniqueness and existence of the risk parity portfolio,

$$\Theta(x) \geq 0, \quad \text{with equality} \quad \iff \mathcal{RC}_i(x) = \mathcal{RC}_j(x) \quad \forall i, j$$

The uniqueness of the risk parity portfolios explains why the other $\theta(x)$ we provided above construct valid risk spreads. In [20] we saw a risk spread measure, Θ where $\theta(x) = \max_{i,j} \{x_i - x_j\}$,

$$\Theta(x) = \max_{i,j} \{\mathcal{RC}_i(x) - \mathcal{RC}_j(x)\}$$

While being a risk spread measure, the above quantity is bounded,

$$\begin{aligned}\min\left\{0, \min_{i,j} \Sigma_{i,j}\right\} \leq (\Sigma x)_i x_i \leq \max_{i,j} \Sigma_{i,j} \\ \Rightarrow \Theta \leq \frac{\max_{i,j} \Sigma_{i,j} - \min\{0, \min_{i,j} \Sigma_{i,j}\}}{\sqrt{x^\top \Sigma x}} \\ \leq \frac{\max_{i,j} \Sigma_{i,j} - \min\{0, \min_{i,j} \Sigma_{i,j}\}}{\mathcal{V}^*}\end{aligned}$$

Here, $\mathcal{V}^*$ is optimal value of the minimum volatility problem,

$$\begin{aligned}\min_x \quad & \sqrt{x^\top \Sigma x} \\ \text{s.t.} \sum_{i=1}^n & x_i = 1\end{aligned}$$

Note that since we assumed $\Sigma \succ 0$, the optimal value $\mathcal{V}^* = \sqrt{x^{*T} \Sigma x^*} > 0$. The optimal value can be trivially computed. First order optimality conditions imply for lagrange multiplier μ corresponding to $\sum_{i=1}^n x_i = 1$,

$$\begin{aligned}x &= \mu \Sigma^{-1} e \\ \Rightarrow \sum_{i=1}^n x_i &= \mu e^\top \Sigma^{-1} e = 1 \\ \Rightarrow \mu &= \frac{1}{e^\top \Sigma^{-1} e} \\ \Rightarrow \mathcal{V}^* &= \sqrt{\mu e^\top (\Sigma^{-1} \Sigma) \mu \Sigma^{-1} e} = \sqrt{\frac{1}{\sum_{i,j} \Sigma^{-1}_{i,j}}}\end{aligned}$$

We can construct instances where this bound is tight. We define tightness as: $\exists \Sigma$ under which an optimal algorithm $\mathcal{A}$ achieves $\forall \epsilon \in \mathbb{R}_+$,

$$\Theta > \frac{\max_{i,j} \Sigma_{i,j} - \min\{0, \min_{i,j} \Sigma_{i,j}\}}{\mathcal{V}^*} - \epsilon$$

Simply meaning for certain instances of Σ we cannot bound the risk spread better than $\frac{\max_{i,j} \Sigma_{i,j} - \min\{0, \min_{i,j} \Sigma_{i,j}\}}{\mathcal{V}^*}$

To show this, consider $\Sigma' = \begin{bmatrix} 1 & 1-\delta \\ 1-\delta & 1 \end{bmatrix} \succ 0$, and $r = (r_1, r_2), \quad r_2 > r_1$. If algorithm $\mathcal{A}$ has to find a portfolio (x, y) that maximizes return subject to $y \leq \delta$. It is easily seen that $(x, y) = (1-\delta, \delta)$, its risk spread in the limit $\delta \to 0$ is given by,

$$\lim_{\delta \to 0} \Theta = \lim_{\delta \to 0} \left| \frac{(1-\delta)^3 - (1-\delta)^2 \delta - \delta^2}{\sqrt{(1-\delta)^3 + \delta(1-\delta)^2 + \delta^2}} \right| = 1$$

Now, $\mathcal{V}^*$ is the optimal value of the problem,

$$\begin{aligned}\min_x \quad & \sqrt{(x+y)^2 - 2\delta xy} \\ \text{s.t.} \quad & x + y = 1\end{aligned}$$

which is clearly equivalent to the problem with the objective squared,

$$\begin{aligned}\min_{x,y} \quad & 1 - 2\delta x(1-x) \\ \text{s.t.} \quad & x \in \mathbb{R}\end{aligned}$$

Hence, the minimum occurs at $x = \frac{1}{2}$, therefore $\mathcal{V}^* = \sqrt{1 - \frac{\delta}{2}}$. Now for our bound on Θ,

$$\lim_{\delta \to 0} \frac{\max_{i,j} \Sigma'_{i,j} - \min\left\{0, \min_{i,j} \Sigma'_{i,j}\right\}}{\mathcal{V}^*} = \lim_{\delta \to 0} \left(\frac{1}{\sqrt{1 - \delta/2}} \right) = 1$$

Hence,

$$\lim_{\delta \to 0} \Theta = \lim_{\delta \to 0} \frac{\max_{i,j} \Sigma'_{i,j} - \min\left\{0, \min_{i,j} \Sigma'_{i,j}\right\}}{\mathcal{V}^*} = 1$$

While the above example is obscure, it only serves to prove the tightness of our naive bound. While the paper bounds the above spread much better than this naive bound to achieve near optimal risk budgeting for higher target returns, no effort is made to optimize this quantity directly. In this paper, we choose appropriate risk spread metrics and aim to minimize them to achieve risk diversified portfolios for higher target returns. In other words, we aim to solve the optimization problem,

$$\min_{x \in \mathcal{P}^R} \quad \Theta(x)$$

where,

$$\mathcal{P}_R = \left\{x : \quad r^\top x \geq R, \quad \mathbf{1}^\top x = 1, \quad x \geq 0\right\}$$

Another example of such a spread metric as mentioned before is the `risk ratio` in our portfolio, where we set $\theta(x) = \max_{i,j} \frac{x_i}{x_j}$

$$\Theta^R = \max_{i,j} \frac{\mathcal{RC}_i(x)}{\mathcal{RC}_j(x)}$$

It can be easily seen risk ratio is a spread metric. This is because

$$\Theta^R = \max_{i,j} \frac{\mathcal{RC}_i(x)}{\mathcal{RC}_j(x)} \geq 1$$

with equality for risk parity. Moreover, at equality we have risk parity as the contributions are all equal, and the uniqueness of the minimum follows from the uniqueness of risk parity. Connecting this to our example above, instead of risk parity if we solve the following optimization problem that minimizes risk ratio,

$$\begin{aligned} \min_x \max_{i,j} \frac{\mathcal{RC}_i(x)}{\mathcal{RC}_j(x)}, &\quad i \in [2], \quad j \in [2] \\ \text{s.t.} \quad & r^\top x \geq 2r_f \\ & \mathbf{1}^\top x = 1 \\ & x \geq 0 \end{aligned}$$

We can now rewrite this problem, using our expressions for risk contributions as,

$$\begin{aligned} \min_x \max_{i,j} \quad \frac{\sigma_i x_i}{\sigma_j x_j}, &\quad i \in [2], \quad j \in [2] \\ \text{s.t.} \quad & r^\top x \geq 2r_f \\ & \mathbf{1}^\top x = 1 \\ & x \geq 0 \end{aligned}$$

We now get a portfolio that tries to squeeze the risk contributions, to make the risk ratio as small as possible, hence diversifying our risk contributions. The above problem is equivalent to

$$\begin{aligned} \min_x \quad & \frac{t}{u} \\ \text{s.t.} \quad & t \geq \sigma_i x_i \quad, \forall i \in [2] \\ & u \leq \sigma_j x_j \quad, \forall j \in [2] \\ & r^\top x \geq 2r_f \\ & \mathbf{1}^\top x = 1 \\ & x \geq 0 \end{aligned}$$

Connecting this to our example above, instead of risk parity if we solve the following optimization problem that minimizes risk ratio:

$$\begin{aligned} \min_x \max_{i,j} \quad & \frac{\sigma_i x_i}{\sigma_j x_j}, \quad i \in [2], \quad j \in [2] \\ \text{s.t.} \quad & r^\top x \geq 2r_f, \\ & \mathbf{1}^\top x = 1, \\ & x \geq 0 \end{aligned}$$

This can be rewritten equivalently by introducing auxiliary variables t and u, which serve as upper and lower bounds for $\sigma_i x_i$ and $\sigma_j x_j$, respectively. This leads to the reformulated problem:

$$\begin{aligned} \min_{x,t,u} \quad & \frac{t}{u} \\ \text{s.t.} \quad & t \geq \sigma_i x_i, \quad \forall i \in [2], \\ & u \leq \sigma_j x_j, \quad \forall j \in [2], \\ & u \geq 0, \\ & r^\top x \geq 2r_f, \\ & \mathbf{1}^\top x = 1, \\ & x \geq 0. \end{aligned}$$

The equivalence follows from the fact that minimizing the worst-case ratio $\max_{i,j}(\sigma_i x_i/\sigma_j x_j)$ is equivalent to minimizing t/u while ensuring that t and u properly bound the respective terms. Now ,if we consider the following change of variables,

$$\frac{t}{u} \to y, \quad \frac{x}{u} \to z, \quad \frac{1}{u} \to v$$

and,

$$\frac{z}{v} \to x, \quad \frac{y}{v} \to t, \quad \frac{1}{v} \to u$$

We get the following `equivalent` linear program,

$$\begin{aligned} \min_x \quad & y \\ \text{s.t.} \quad & y \geq \sigma_i z_i \quad , \forall i \in [2] \\ & 1 \leq \sigma_j z_j \quad , \forall j \in [2] \\ & r^\top z \geq 2r_f v \\ & \mathbf{1}^\top z = v \\ & z \geq 0 \end{aligned}$$

Here our spread metric was, $\Theta(x) = \max_{i,j} \frac{\mathcal{RC}_i(x)}{\mathcal{RC}_j(x)}$ with the optimal solution being $x^* = \frac{z^*}{v^*}$. Here it is easily seen that, if our return target is less than or equal to the return of risk parity, $\min_{x \in \mathcal{P}^{r*}} \Theta(x) = 1$ with optimal $x = x^*$. For a larger target, our spread metric tries to squeeze the ratio between the maximum and the minimum risk, which is one way to achieve diversification. More so we can generalize this to more than two assets, giving us a solution for optimal risk budgeting involving target returns (`ORBIT`), in the uncorrelated case. This is done by the following linear program-

$$\begin{aligned} \min_x \quad & y \\ \text{s.t.} \quad & y \geq \sigma_i z_i \quad , \forall i \in [n] \\ & 1 \leq \sigma_j z_j \quad , \forall j \in [n] \\ & r^\top z \geq 2r_f v \\ & \mathbf{1}^\top z = v \\ & z \geq 0 \end{aligned}$$

We now do the same for the following for another spread metric, least invested risk ratio, Θ^L defined by,

$$\Theta^L(x) = -\min_j \frac{\mathcal{RC}_j(x)}{\sum_{i=1}^{n} \mathcal{RC}_i(x)}$$

This spread metric tries to push the least invested risk as high as possible to encourage risk diversification. It easily follows that this is a spread metric. Using the same problem data as above, i.e. diagonal Σ, r and target R, we can maximize the negative of Θ^L. The optimization problem is given by,

$$\begin{aligned} \max_x \min_i \quad & \frac{\sigma_i^2 x_i^2}{\sum_{i=1}^n \sigma_i^2 x_i^2} \\ \text{s.t.} \quad & r^\top x \geq 2r_f \\ & \mathbf{1}^\top x = 1 \\ & x \geq 0 \end{aligned}$$

Using the same trick as we used above, this is equivalent to,

$$\begin{aligned} \max_x \quad & \frac{u^2}{\sum_{i=1}^n \sigma_i^2 x_i^2} \\ \text{s.t.} \quad & u \leq \sigma_i x_i \\ & r^\top x \geq 2r_f \\ & \mathbf{1}^\top x = 1 \\ & x \geq 0 \end{aligned}$$

Now if we consider the following change of variables,

$$\frac{x}{u} \to z, \quad \frac{1}{u} \to v$$

and,

$$\frac{z}{v} \to x, \quad \frac{1}{v} \to u$$

we get the following `equivalent` QP for maximizing least invested risk,

$$\begin{aligned} \min_z \quad & z^\top \Sigma z \\ \text{s.t.} \quad & 1 \leq \sigma_i z_i \\ & r^\top z \geq 2r_f v \\ & \mathbf{1}^\top z = v \\ & z \geq 0 \end{aligned}$$

The following sections tries to solve risk diversification problem with two spread metrics, i.e. least invested risk ratio $\Theta^L(x) = -\min_j \frac{\mathcal{RC}_j(x)}{\sum_{i=1}^n \mathcal{RC}_i(x)}$ and worst case risk ratio $\Theta^R(x) = \max_{i,j} \frac{\mathcal{RC}_i(x)}{\mathcal{RC}_j(x)}$. We call the former algorithm `LIRA` and the latter `ORBIT`. We already saw that there is an efficient way (polynomial time) of solving `ORBIT` in the uncorrelated case. In the following sections we highlight the complexities of the general cases and provide exact and approximation algorithms (with bounds depending only on problem data i.e. Σ and target return) for solving these problems. Next we provide definitions that we will used repeatedly in the defining our algorithms and the proof of their performance guarantees.

2 Definitions

We introduce the following key definitions used throughout this paper:

- `Risk Measure`:

$$\mathcal{R}(x) = \sqrt{x^\top \Sigma x}$$

where $\mathcal{R}(x)$ represents the risk measure applied to the portfolio weights x.

- `Risk Contribution`:

$$\mathcal{RC}_i(x) = x_i \frac{\partial \mathcal{R}(x)}{\partial x}$$

which defines the risk contribution of the ith asset in a portfolio weighted by x, consistent with the risk parity framework.

- `Return-Constrained Portfolio Set` $\mathcal{P}_R$:

$$\mathcal{P}_R = \left\{ x \,\middle|\, r^\top x \geq R, \quad \mathbf{1}^\top x = 1, \quad x \geq 0 \right\}$$

representing the set of portfolio allocations that satisfy a minimum return requirement while ensuring full allocation and non-negativity constraints.

- `Attainable Return-Risk Set` $\mathfrak{R}^{r,\gamma}$:

$$\mathfrak{R}^{r,\gamma} = \left\{ R \,\middle|\, \exists x > 0 \text{ such that } r^\top x \geq R \right\}$$

characterizing the set of attainable returns R that can be achieved while satisfying the given risk constraints.

- `Risk Spread Metric` Θ:
A risk spread measure $\Theta : \Delta \to \mathbb{R}$ is a continuous function on the domain:

$$\mathcal{X} = \left\{ x \middle| \quad \mathcal{RC}_j(x) > 0, \quad \forall j \right\}$$

s.t. risk parity parity portfolio is a unique solution to the problem

$$\min_{x \in \mathcal{P}^{R_p}} \quad \Theta(x)$$

3 Algorithms for Optimal Risk Budgeting Involving Targets (`ORBIT`)

We now provide algorithms for performing diversified risk budgeting with target returns. In the algorithms we provide, we do not allow short selling, i.e., $x \geq 0$. This helps facilitate the construction of efficient algorithms to find desired portfolios.

3.1 Least Invested Risk Amplifier-`LIRA`

Our first algorithm, `LIRA` aims to maximize the following quantity,

$$\max_{x \in \mathcal{P}^R} \qquad \underbrace{\min_j \frac{\mathcal{RC}_j(x)}{\sum_{i=1}^{n} \mathcal{RC}_i(x)}}_{\texttt{Least Investment Risk Ratio}, \quad \Theta^L}$$

The quantity represents the proportion of risk contributed by the least risky asset, maximizing this quantity represents diversifying our risk contributions. The problem can be rewritten as

$$\max_{x \in \mathcal{P}^R} \min_j \frac{x_j (\Sigma x)_j}{x^\top \Sigma x}$$

Our numerator is a quadratic form $x^\top \Sigma^{(j)} x$, where

$$\Sigma^{(i)}_{k,l} = \begin{cases} \frac{\Sigma_{i,l}}{2} & k = i, \quad l \neq i \\ \frac{\Sigma_{i,l}}{2} & l = i, \quad k \neq i \\ \Sigma_{i,i} & i = k = l \\ 0 & \text{o.w.} \end{cases}$$

We show that this symmetric matrix has a negative eigenvalue, making our problem potentially complicated. However, we have the following theorem,

Theorem 2. *(`LIRA`) There exists an efficient algorithm $\mathcal{A}$ that finds x^* solving $\max_{x \in \mathcal{P}^R} \min_j \frac{\mathcal{RC}_j(x)}{\sum_{i=1}^n \mathcal{RC}_i(x)}$, for risk measure $R(x) = \sqrt{x^\top \Sigma x}$*

Here (and in subsequent algorithms), we define an `efficient` algorithm as the existence of a convex program whose solution provides us with the desired portfolio. Using the result of Theorem 1, we have the following algorithm:

Algorithm 1 `LIRA`

`Input:` $\Sigma \succ 0$ and return target R

1. Solve the `convex` problem

$$\begin{aligned} \min \quad & z^\top \Sigma z \\ \text{s.t.} \quad & \left\| \begin{pmatrix} 1 \\ z_j \\ (\Sigma z)_j \end{pmatrix} \right\|_2 \leq z_j + (\Sigma z)_j \\ & r^\top z = w \\ & \mathbf{1}^\top z = w \end{aligned}$$

and obtain optimal solution (z^*, w^*)

`return` $x^* = \frac{z^*}{w^*}$

We now have the following corollary:

Corollary 3. *Let $r_{\texttt{ERC}}$ be return of an `ERC` portfolio, with Σ. If $R \leq r_{\texttt{ERC}}$, and x^* is an optimal solution for `LIRA`, then $x^* = x_p$, where x_p is an `ERC` portfolio given Σ.*

This shows that if our return target is less than or equal to the return of an `ERC` portfolio, `LIRA` returns an `ERC` portfolio. For greater return targets, `LIRA` will push $\min_j \frac{\mathcal{RC}_j(x)}{\sum_{i=1}^n \mathcal{RC}_i(x)}$ as close to $\frac{1}{n}$ as possible.

3.2 Overcoming the Pitfalls of `LIRA`: `ORBIT`

While `LIRA` provides a way to diversify our risks with a return target, it only focuses on the least risk in our portfolio, which can make the larger risks potentially concentrated. As a motivation, we plot $\max_j \frac{\mathcal{RC}_j(x)}{\sum_{i=1}^n \mathcal{RC}_i(x)}$ and $\min_j \frac{\mathcal{RC}_j(x)}{\sum_{i=1}^n \mathcal{RC}_i(x)}$ for `LIRA` with increasing target returns and the new algorithm we introduce in this section ϵ-`ORBIT` in Figure 2 (here, $n = 6$)

As we can see, `LIRA` is not able to control $\max_j \frac{\mathcal{RC}_j(x)}{\sum_{i=1}^n \mathcal{RC}_i(x)}$ as the risk appetite increases. To fix this, we must incorporate the maximum risk in our portfolio into the spread metric. To this extent, define the `ORBIT` problem as,

$$\min_{x \in \mathcal{P}^R} \quad \underbrace{\max_{i,j} \frac{\mathcal{RC}_i(x)}{\mathcal{RC}_j(x)}}_{\texttt{Total Risk Spread Ratio},\quad \Theta^R}$$

Substituting expressions for risk contributions, we get the following optimization problem,

$$\min_{x \in \mathcal{P}^R} \max_{i,j} \frac{x_i(\Sigma x)_i}{x_j(\Sigma x)_j} = \min_{x \in \mathcal{P}^R} \max_{i,j} \frac{x^\top \Sigma^{(i)} x}{x^\top \Sigma^{(j)} x} \tag{ORBIT}$$

As discussed above, our objective is challenging due to the non-convex nature of the quadratic forms. To address this, we seek approximations to

$$x^* \in \operatorname*{argmin}_{x \in \mathcal{P}^R} \max_{i,j} \frac{x_i(\Sigma x)_i}{x_j(\Sigma x)_j}$$

Specifically, we derive a constant-factor approximation for `ORBIT` by constructing an efficient algorithm— a second-order conic program (`SOCP`)—to find x_* such that

$$\max_{i,j} \frac{\mathcal{RC}_i(x_*)}{\mathcal{RC}_j(x_*)} \leq \alpha(\Sigma, R) \left(\max_{i,j} \frac{\mathcal{RC}_i(x^*)}{\mathcal{RC}_j(x^*)} \right), \quad \alpha \geq 1$$

where x^* is an optimal solution to `ORBIT`. Next, we bound the value of α in terms of the covariance matrix, Σ and our return target, R. We then test the magnitude of $\alpha(\Sigma, R)$ on real world and simulated data. Before we move onto specifying our algorithm, we mention two things that motivate our work. Firstly,

$$\max_{i,j} \frac{\mathcal{RC}_i(x)}{\mathcal{RC}_j(x)} \geq 1, \quad \forall x \in \mathcal{P}^R$$

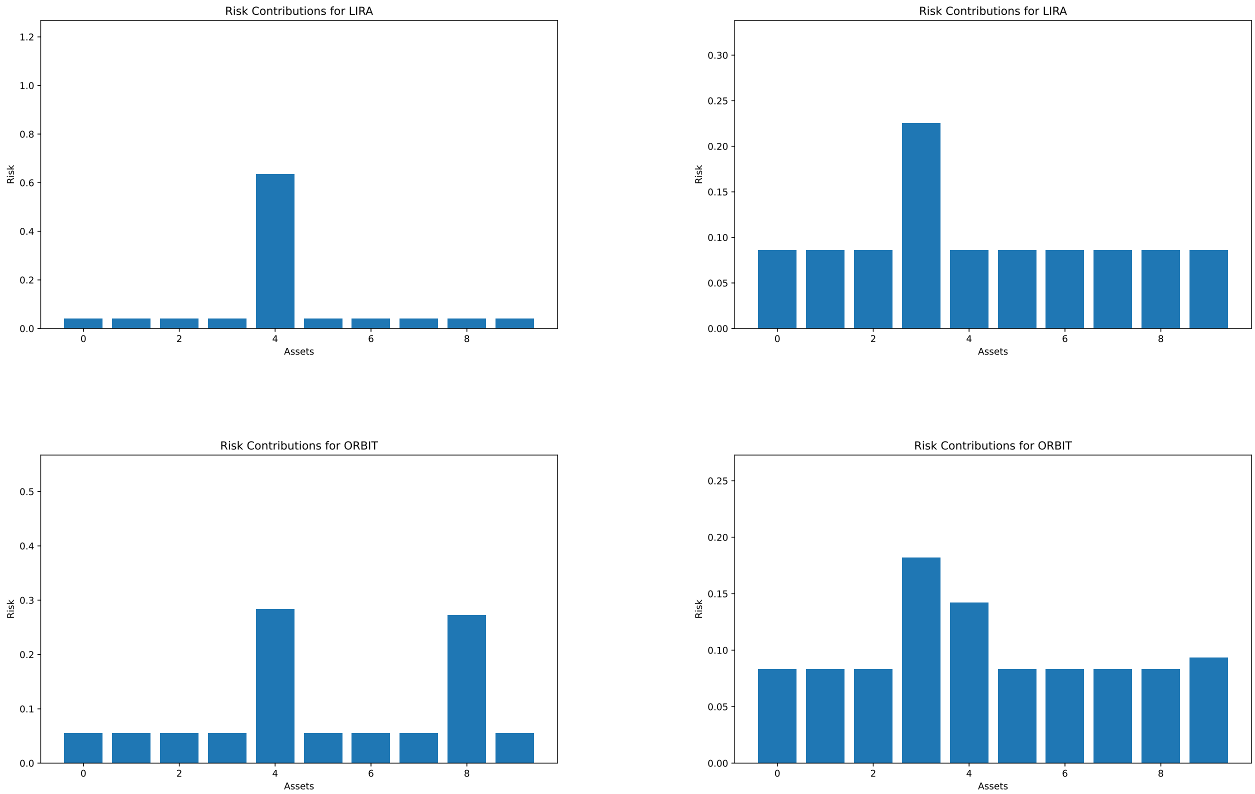

Figure 1: Risk Contributions for high and moderate risk (left and right) appetite for `LIRA` and ϵ-`ORBIT` respectively (top and bottom).

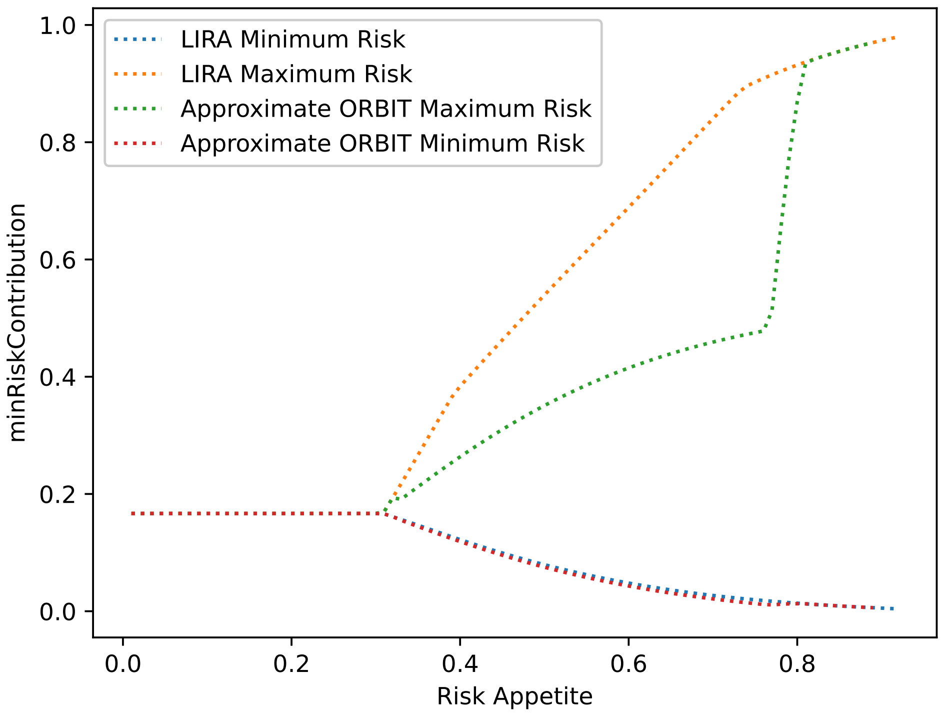

Figure 2: Spread of Risks for `LIRA` and ϵ-`ORBIT` with increasing targets, here risk appetite is γ, defined by $R = \gamma \max_i r_i + (1 - \gamma) \min_i r_i$

And moreover,

$$\max_{i,j} \frac{\mathcal{RC}_i(x)}{\mathcal{RC}_j(x)} = 1, \quad \forall R \leq r_{\mathtt{ERC}}$$

This means that if we have a return target less than $r_{\mathtt{ERC}}$ it is optimal (w.r.t. Total Risk Spread Ratio), to pick $x_{\mathtt{ERC}}$. So not only do we provide an approximation algorithm to `ORBIT`, we also provide bounds to $\max_{i,j} \frac{\mathcal{RC}_i(x)}{\mathcal{RC}_j(x)}$ in the regime, $R \leq r_{\texttt{ERC}}$

3.2.1 Constant Factor Approximation to `ORBIT`: ϵ-`ORBIT`

As previously motivated, we now develop an approximation algorithm for `ORBIT`. The following theorem formalizes our result:

Theorem 4. *let x_* be an optimal solution to ϵ-`ORBIT` with objective value Θ_*, x^* be an optimal solution to `ORBIT` with objective value Θ^*. Moreover, assume that,*

$$x^{*\top}x^* \leq \mathbb{E}\left[x^\top x\right], \quad x \sim Dir(1,\dots,1)$$

Then x_* is a $\left\{1 + \left(\sqrt{\sum_k \Sigma_{k,i}^2} - \Sigma_{i,i}\right)\sum_{i,j}\Sigma_{i,j}^{-1}\right\}$ *approximation to `ORBIT`, i.e.*

$$\Theta(x_*) \leq \alpha(\Sigma, R)\Theta(x^*)$$

where,

$$\alpha(\Sigma, R) = \left\{1 + \left(\sqrt{\sum_k \Sigma_{k,i}^2} - \Sigma_{i,i}\right)\sum_{i,j}\Sigma_{i,j}^{-1}\right\}$$

Where ϵ-`ORBIT` is our approximation algorithm to `ORBIT`. Moreover, ϵ-`ORBIT` is an efficient algorithm and is an SOCP that can be solved easily by modern optimizers. Our algorithm corresponding to this bound is given by,

Algorithm 2 ϵ-`ORBIT`

`Input:` $\Sigma \succ 0$ and return target R

1. Compute $\lambda = -\max_i \left\{\frac{\sqrt{\sum_k \Sigma_{k,i}^2} - \Sigma_{i,i}}{2}\right\}$
2. Solve the `Second Order Conic Program (SOCP)`

$$\begin{aligned}
\min \quad & y \\
\text{s.t.} \quad & \left\| \begin{pmatrix} v \\ z_j \\ (\Sigma z)_j \end{pmatrix} \right\|_2 \leq z_j + (\Sigma z)_j \\
& \left\| \begin{pmatrix} 1 \\ y \\ v \end{pmatrix} \right\|_2 \leq y + v \\
& 1 \geq \left\| (\Sigma^{(i)} - \lambda I)^{\frac{1}{2}} z \right\|_2 \\
& r^\top z = w \\
& \mathbf{1}^\top z = w
\end{aligned}$$

and obtain optimal solution (z^*, w^*, y^*, v^*)

`return` $x^* = \frac{z^*}{w^*}$

Corollary 5. *(Proximity of ϵ-`ORBIT` to risk parity) If $R \leq r_{\texttt{ERC}}$, x^* be an optimal solution to ϵ-ORBIT and x_p be a risk parity solution and γ chosen such that x_p feasible for ϵ-ORBIT, then*

$$\max_{i,j} \frac{x_i^*(\Sigma x)_i}{x_j^*(\Sigma x^*)_j} - 1 \leq |\lambda| \left(n\frac{x_p^\top x_p}{x_p^\top \Sigma x_p} - \frac{1}{\lambda^*}\right)$$

This corollary is very important as we don't have a reduction corollary to risk parity, unlike the one in `LIRA`. We now present a corollary that shows the proximity of ϵ-`ORBIT` to risk parity.

Discussing our bound $\alpha(\Sigma, R)$

In this section we make efforts to improve our bound $\alpha(\Sigma, R)$ and to provide its behavior in the case of weakly correlated assets. From the proof of theorem 1, it will easily follow that,

$$\begin{aligned}\alpha(\Sigma, R) &\leq \left\{ 1 + \left(\frac{\max_i \left(\sqrt{\sum_k \Sigma_{k,i}^2} - \Sigma_{i,i} \right)}{2} \right) \frac{x^{*\top} x^*}{\max_i x_i^* (\Sigma x^*)_i} \right\} \\ &\leq \left\{ 1 + \left(\frac{\max_i \left(\sqrt{\sum_k \Sigma_{k,i}^2} - \Sigma_{i,i} \right)}{2} \right) \frac{x^{*\top} x^*}{\underbrace{\min_{x \in \mathcal{P}^R} \max_i x_i (\Sigma x)_i}_{\mathbf{P}}} \right\}\end{aligned}$$

Where x^* is optimal solution to **ORBIT**. We will see in the proof of theorem 1 that the above bound is a lot better than the bound for α stated in theorem 1. To compute this bound however, we need to be able to solve the following problem **P**

$$\min_{x \in \mathcal{P}^R} \max_i x_i (\Sigma x)_i$$

The above problem is not convex at first sight, however we can find a lower bound by solving the following problem,

$$\begin{aligned} \min \quad & \gamma \\ \text{s.t.} \quad & \gamma \geq \left\| (\Sigma^{(i)} - \lambda I)^{\frac{1}{2}} x \right\|_2 \\ & r^\top x = R \\ & \mathbf{1}^\top R = 1 \end{aligned} \tag{$\mathbf{P'}$}$$

The problem **P'** gives us a solution x, whose properties are described in the following lemma,

Lemma 6. *Let x be an optimal solution to above problem **P'**, let x^* be an optimal solution **P**. Define $\epsilon = \lambda \|x - x^*\|_2$, then we have that*

$$\min_{x \in \mathcal{P}^R} \max_i x_i (\Sigma x)_i \leq \min_{x^* \in \mathcal{P}^R} \max_i x_i^* (\Sigma x^*)_i + \epsilon$$

That is, the optimality gap is given by ϵ.

Now if ϵ is small enough, it is seen that the revised bound is much better than the bounds stated in theorem 1. Moreover, we highlight the effectiveness of our bound on real world data, and provide monte carlo simulations of our unsimplified bound on the simplex where portfolios are drawn from in the appendix **A.1**. This will be discussed in the proofs section. Now, we analyze the behavior of our bound when our assets are weakly correlated, i.e., the off-diagonal terms on the covariance matrix have lower magnitude. It is easily noticed that our bound for ϵ- `ORBIT` gets better as the covariances go to zero. This is because

$$\begin{aligned} \max_i \left\{ \sqrt{\sum_k \Sigma_{k,i}^2} - \Sigma_{i,i} \right\} &= \sqrt{\sum_k \Sigma_{k,I}^2} - \Sigma_{I,I} \\ &\leq \underbrace{\Sigma_{I,I} + \sqrt{\sum_{k \neq I} \Sigma_{k,I}^2}}_{\because \sqrt{|a|+|b|} \leq \sqrt{|a|} + \sqrt{|b|}} - \Sigma_{I,I} \\ &\leq \max_j \left\{ \sqrt{\sum_{k \neq j} \Sigma_{k,j}^2} \right\} \\ &\leq \sqrt{n} \max_{i,j,i \neq j} |\Sigma_{i,j}| \end{aligned}$$

Tightness in the diagonal case is expected as we saw in our preliminary example in the introduction- `ORBIT` reduces to a linear programming (`LP`) in the uncorrelated case. Approximate reduction is also seen around a neighborhood of small correlation values. More formally, we have the following corollary:

Corollary 7. *Let $\mathbb{V}$ be diagonal matrix of volatilities of Σ and define Σ_γ to be*

$$\Sigma_\gamma = \gamma\mathbb{V} + (1-\gamma)\Sigma$$

Then we have that,

$$\lim_{\gamma\to 1} \quad \alpha(\Sigma_\gamma, R) = 1$$

Moreover, $\exists\gamma'$ such that, $\forall\gamma \geq \gamma'$,

$$\alpha(\Sigma, R) \leq O\left(1 + \frac{(1-\gamma)K}{L-(1-\gamma)M}\right)$$

for K, L, M that depend on Σ. This also characterizes the rate at which $\alpha(\Sigma, R)$ goes to 1.

The above corollary formalizes the fact that as the strength of the correlations go to zero, our bound on ϵ- `ORBIT` becomes tighter. We will use real world data to test the effectiveness of ϵ-`ORBIT` for return targets more than that of risk parity. Another thing to note is that our bound on α is so for independent on R, the return target. As we will see in the proof of the above theorem, our bound can slightly be modified,

$$\alpha(\Sigma, R) = \left\{ 1 + \frac{n \max_i \left(\sqrt{\sum_k \Sigma_{k,i}^2} - \Sigma_{i,i}\right)}{2\left(\frac{1}{\sum_{i,j}\Sigma_{i,j}^{-1}} + h(R)\right)} \right\}$$

where $h(R)$ is a monotonic increasing function in R. So far we give algorithms for risk diversification while fixing portfolio volatility to be our risk measure.

3.2.2 CVaR Budgeting Involving Targets (`CVaR-BIT`)

We now move onto considering a different risk measure, conditional value at risk- `CVaR`. We provide our next algorithm for risk diversification using this risk measure- `CVaR` budgeting involving targets, `CVaR-BIT`. Let $F_r(z) = \mathbb{P}(r \leq z)$, denote the cumulative distribution function of portfolio return r. We define value at risk and conditional value at risk at level α to be:

$$\text{VaR}_\alpha = \min\{z : F_r(z) \geq \alpha\}$$

and

$$\text{CVaR}_\alpha = \mathbb{E}\left[r | r \geq \text{VaR}_\alpha\right]$$

Now assuming normal returns, i.e., $r \sim \mathcal{N}(\Sigma, \mu)$, from [17] we have that,

$$\text{CVaR}_\alpha(x) = \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}\sqrt{x^\top \Sigma x} - r^\top x$$

The risk contributions for CVaR are given by:

$$\mathcal{RC}_i^{\text{CVaR}}(x) = \frac{\partial \text{CVaR}(x)}{\partial x_i} x_i$$

which implies

$$\mathcal{RC}_i^{\text{CVaR}}(x) = -\mu_i x_i + \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}\frac{x_i(\Sigma x)_i}{\sqrt{x^\top \Sigma x}}$$

We now define our risk spread metric:

$$\begin{aligned}\Theta^{\text{CVaR}}(x) &= \frac{\min_i \mathcal{RC}_i^{\text{CVaR}}(x)}{\text{CVaR}(x)} \\ &= \frac{\min_i \left\{-\mu_i x_i + \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}\frac{x_i(\Sigma x)_i}{\sqrt{x^\top\Sigma x}}\right\}}{\text{CVaR}(x)}\end{aligned}$$

and our problem CVaR budgeting involving risks `CVaR-BIT`,

$$\max_{x\in\mathcal{P}^{R,\gamma}} \frac{\min_i \mathcal{RC}_i^{\text{CVaR}}(x)}{\text{CVaR}(x)} = \max_{x\in\mathcal{P}^{R,\gamma}} \frac{\min_i\left\{-\mu_i x_i + \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}\frac{x_i(\Sigma x)_i}{\sqrt{x^\top\Sigma x}}\right\}}{\text{CVaR}(x)}$$

Here, we use the set $\mathcal{P}^{R,\gamma}$ for analytical tractability. Moreover, it helps us bound our portfolio volatility as it is no longer our risk measure. To motivate our results, let's try to diversify contributions of CVaR using `LIRA`, ϵ-`ORBIT` and our new algorithm that we obtain in this section.

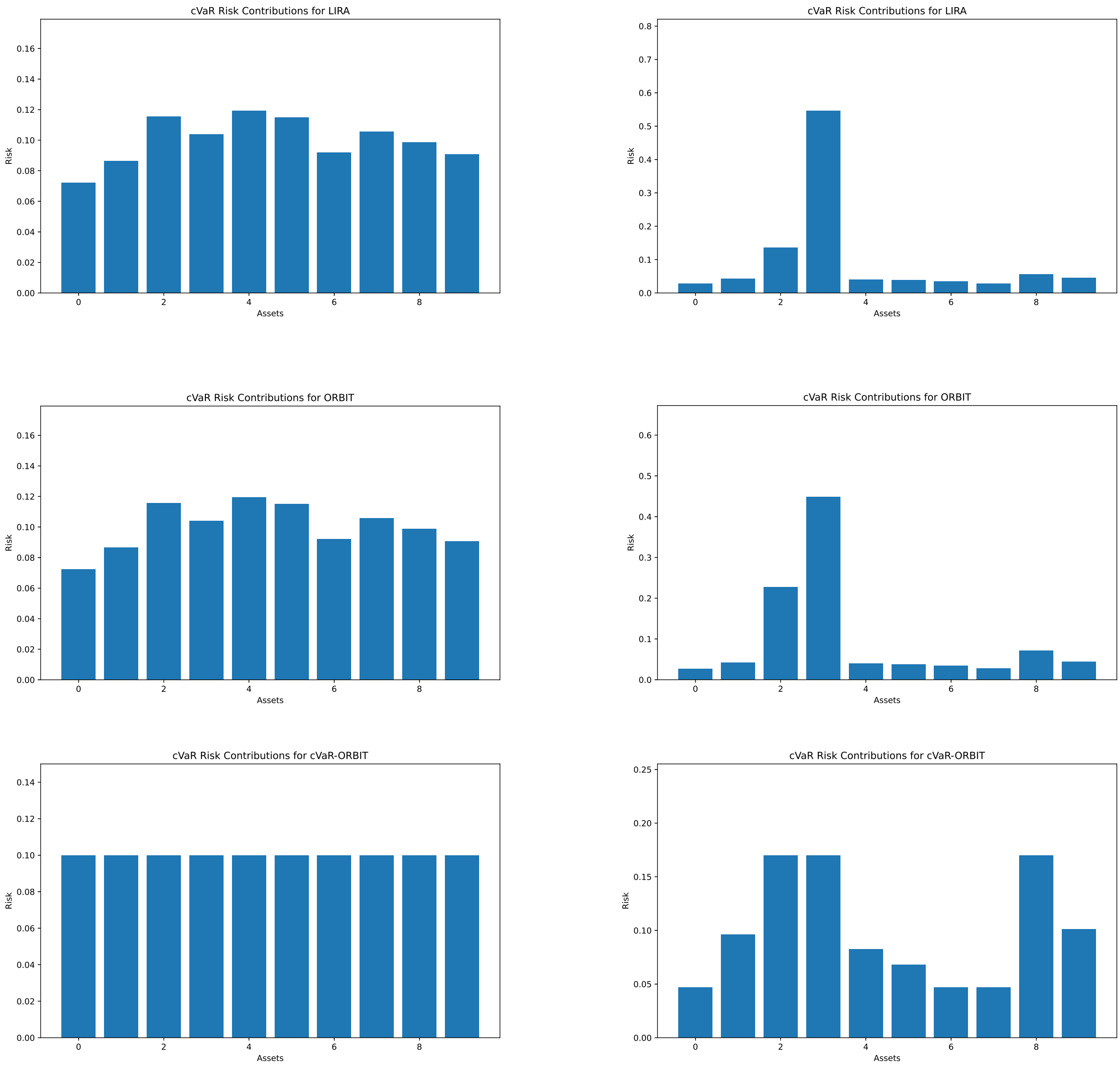

Figure 3: CVaR Risk Contributions for `LIRA`(Top), `ORBIT`(Middle) and `CVaR-ORBIT`(Bottom) for low(left) and high(right) risk appetites, $\gamma = 0.2, 0.8$ respectively

It is clearly seen that while `LIRA` and ϵ-`ORBIT` worked well on volatility as a risk measure, CVaR risk measure diversification seemed to not be this effective. However, the algorithm that we introduce in this section seems to have superior performance in diversifying CVaR. In this vein, we now have the following theorem,

Theorem 8. *Assume that,*

$$\Theta^L(x_L) \geq \frac{(1-\alpha)\max_j r_j \sum_{i,j}\Sigma_{i,j}^{-1}}{\phi(\Phi^{-1}(\alpha))}$$

and that,

$$\max_j r_j \left(\sum_{i,j}\Sigma_{i,j}^{-1}\right) \leq w_\alpha$$

where Θ^L and x_L are spread metric and solution to **LIRA***. Moreover, assume $r \geq 0$. Then there exists an efficient algorithm that finds x_*,*

$$\Theta^{CVaR}(x_*) \geq \epsilon \Theta^{CVaR}(x^*)$$

$$\epsilon = \left(\frac{w_\alpha \sqrt{x_*^\top \Sigma x_*}}{CVaR(x_*)} \right) \left(1 - \frac{SR^*}{w_\alpha} \right) \Theta^{CVar}(x^*)$$

Where SR^* is the optimal value of the Sharpe ratio problem,*

$$\max_{x \in \mathcal{P}^R} \frac{r^\top x}{\sqrt{x^\top \Sigma x}}$$

Here, x^* is optimal solution to CVaR-LIRA. The algorithm to find such an x_* is given by:

Algorithm 3 ϵ-CVaR-BIT

Input: $\Sigma \succ 0$ and return target R

1. Compute $\lambda_- = -\max_i \left\{ \frac{\sqrt{\sum_k \Sigma_{k,i}^2} - \Sigma_{i,i}}{2} \right\}$
2. For a long enough horizon, $\mathcal{H}$, choose $R_j = (r - e \min_{i,t \in \mathcal{H}} r_i^t)_j$
3. Solve the convex problem

$$\begin{aligned} \min_z \; & z^\top \Sigma z \\ \text{s.t.} \quad & u_i \leq \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha} (\Sigma z)_i - r_i \left\| \Sigma^{\frac{1}{2}} z \right\|_2, \quad \forall i \\ & \left\| \begin{pmatrix} u \\ z_j \\ (\Sigma z)_j \end{pmatrix} \right\|_2 \leq z_j + (\Sigma z)_j \\ & r^\top z \geq R w \\ & e^\top z = w \end{aligned} \tag{\textbf{P}}$$

and obtain optimal solution x^*

return x^*

It has been shown in the literature that it is better to consider a heavy-tailed distribution for analyses involving CVaR because

- Returns are not normally distributed in practice
- Effects of CVaR are seen more predominantly with heavy tailed distributions

Previous works suggest using the t-distribution, where CVaR is given by

$$\phi(\Phi^{-1}(\alpha)) \sqrt{x^\top \Sigma x} - r^\top x.$$

Our method can be directly applied to this expression as well. We now give a summary of our algorithms, what risk measure they use and under what framework do they diversify risks (θ) in the following table,

Table 1: Risk Distribution for $1/n$ and MVO

Algorithm	Risk Measure	Diversification Measure θ
LIRA	Volatility	$-\min_i \frac{x_i}{\sum_{j=1}^n x_j}$
ϵ-**ORBIT**	Volatility	$\max_{i,j} \frac{x_i}{x_j}$
CVaR-**LIRA**	cVaR	$-\min_i \frac{x_i}{\sum_{j=1}^n x_j}$

4 Simulations and Numerical Experiments

4.1 Experiments for Comparison with Risk Parity and Evolution of Spread with γ

Here, we increase our risk appetite γ defined as,

$$R = \gamma \min_j r_j + (1-\gamma) \max_j r_j$$

and show the changes in our risk distribution for ϵ- **ORBIT** and **LIRA**. Here we use simulated geometric brownian motion.

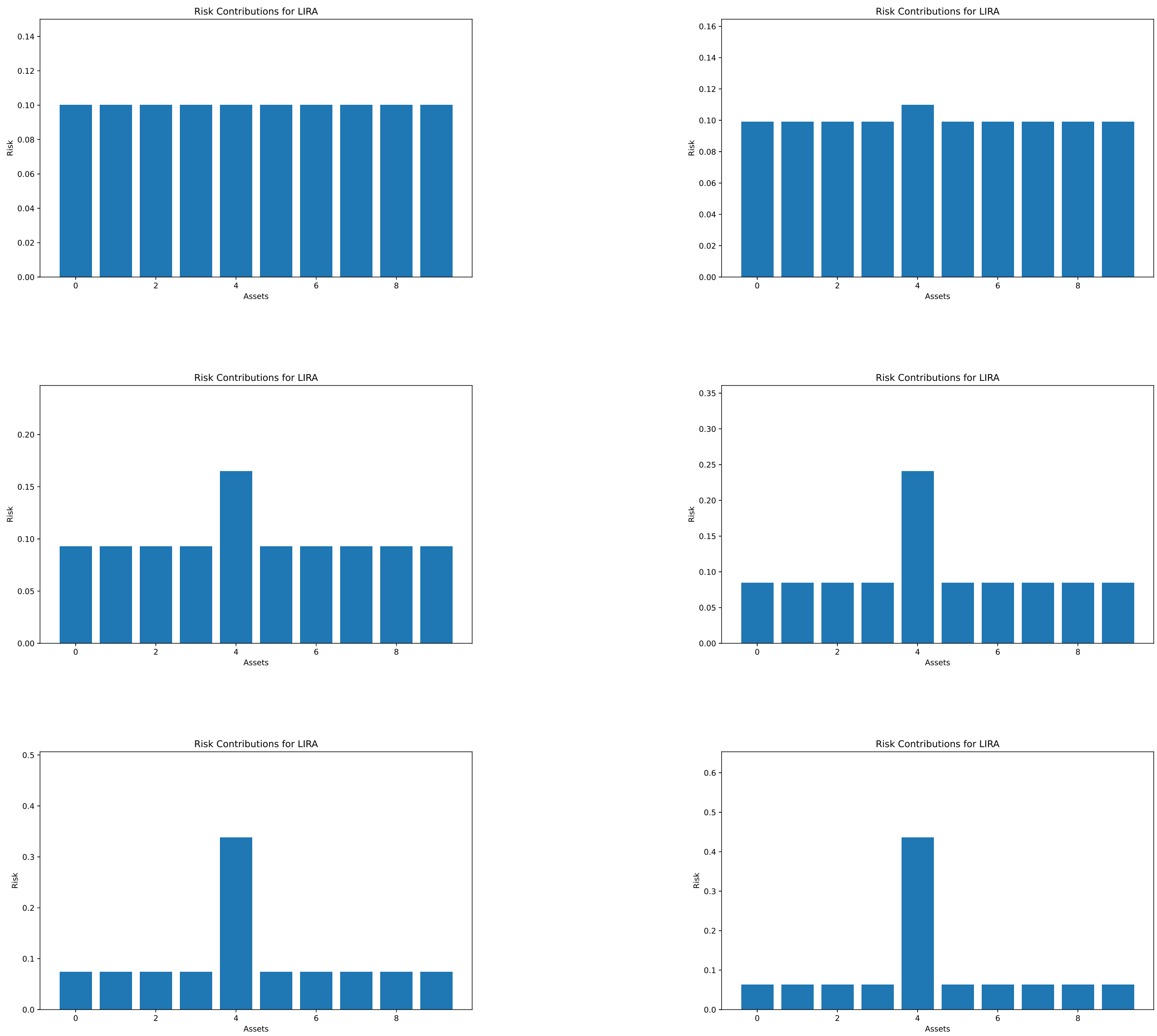

Figure 4: Evolution of risk contributions with increasing values of γ (increasing left to right top to bottom, $\gamma = 0.45, 0.48, 0.51, 0.54, 0.57, 0.6$) for **LIRA**

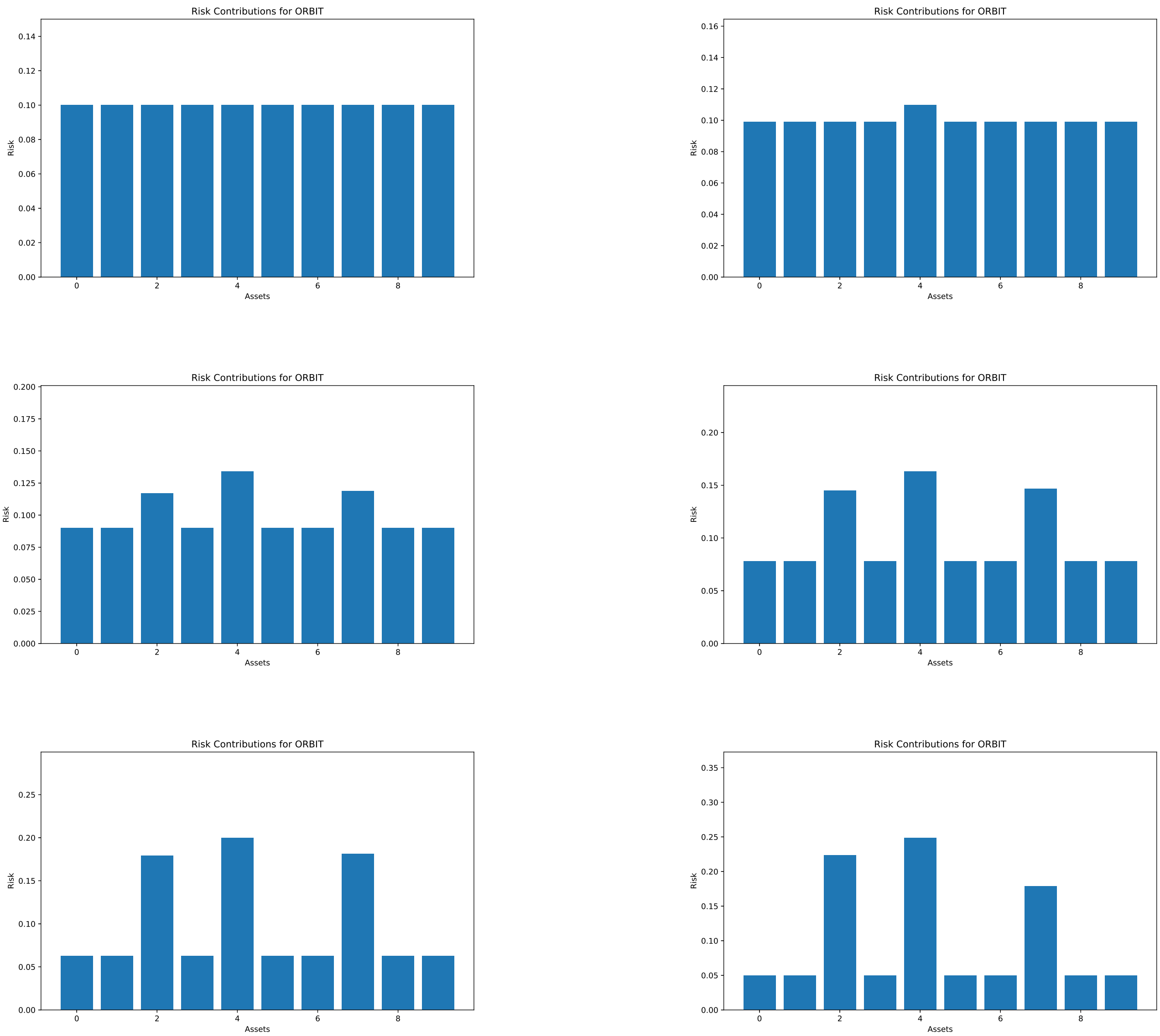

Figure 5: Evolution of risk contributions with increasing values of γ (increasing left to right top to bottom, $\gamma = 0.45, 0.48, 0.51, 0.54, 0.57, 0.6$) for ϵ-**ORBIT**

4.2 Evolution of Risk Contributions and Spread Metrics over Risk Appetite: Our Algorithms and Benchmarks

In this section, we provide evolution of our risk spread metrics for `LIRA` and ϵ-`ORBIT` over risk appetite/ We compute these quantities over simulated geometric Brownian motion with fixed drift and volatility. The benchmarks we consider over our algorithms are, $1/n$, `MVO` with return lower bound R and conic program as given by model A in [11]. Since $1/n$ cannot achieve all possible returns, i.e.

$$R_{1/n} = \sum_{i=1}^{n} r_i$$

and `MVO` suffers from portfolio concentration, i.e. for high values of R it is actually desirable for the algorithm to take some $S \subset [n]$ s.t

$$x_i = 0, \quad \forall i \in S$$

causing our risk ratio to blow up, we do not plot these in the figures below. Plots for these two benchmarks can be found in the Appendix.

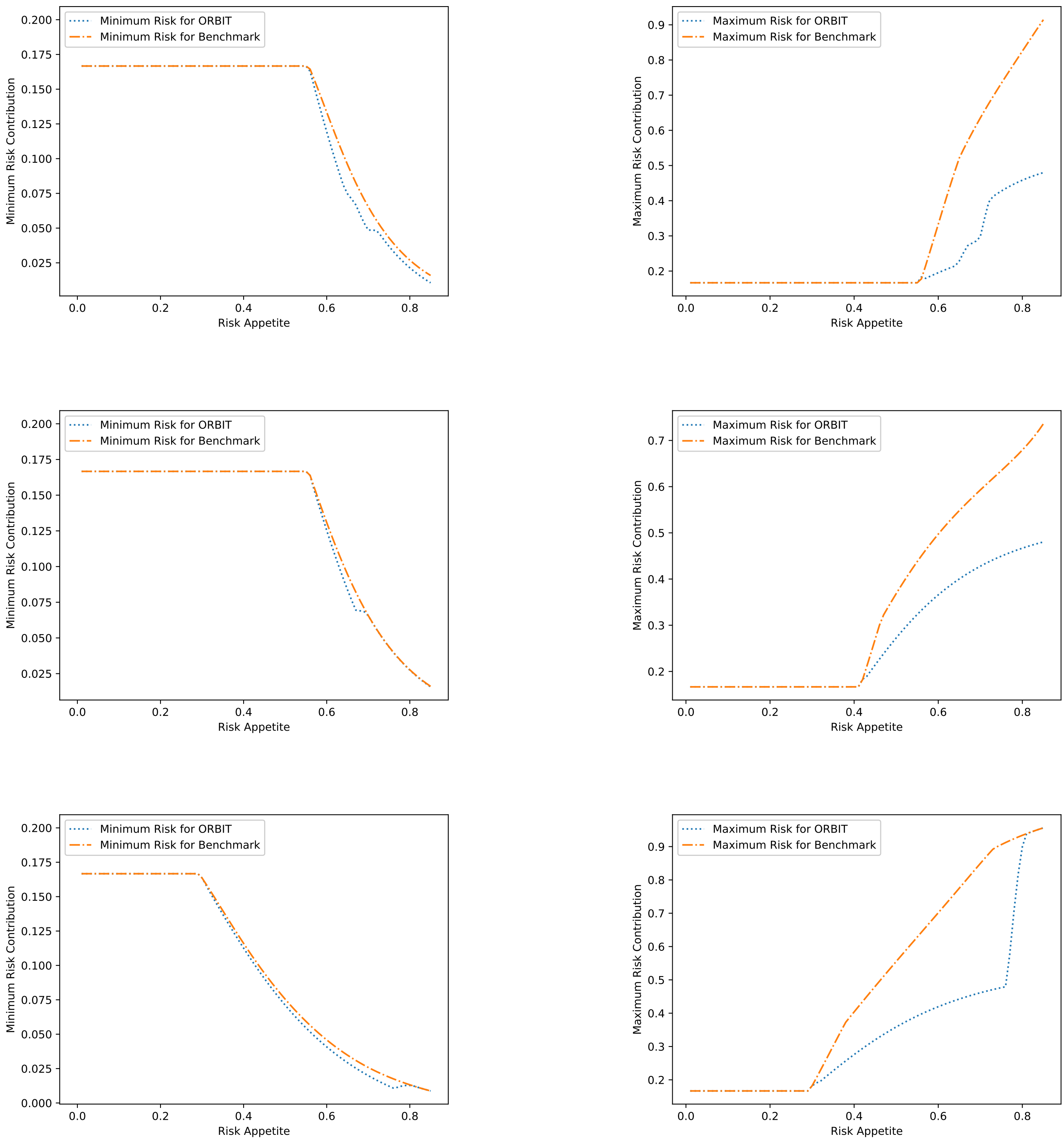

Figure 6: Evolution of $\Theta(x) = \min_j \frac{\mathcal{RC}_j(x)}{\sum_{i=1}^{n} \mathcal{RC}_i}$ and $\max_j \frac{\mathcal{RC}_j(x)}{\sum_{i=1}^{n} \mathcal{RC}_i}$ with γ

We cannot really plot `MVO` and $1/n$ portfolios on these graphs, due to problems discussed in Table 1. We also provide some statistics for these algorithms.

Table 2: Risk Distribution for $1/n$ and `MVO`

Algorithm	Limitations	Description
$1/n$	Return agnostic, cannot achieve a return higher than $\frac{\sum_i r_i}{n}$	Risk Ratio Θ for the three simulations= 4.12, 3.14, 2.78
`MVO`	Risk Ratio blows to infinity due to portfolio concentration	γ_c for the three simulations= 0.73, 0.82, 0.71 (This is the risk appetite for which the risk ratio blows up)

4.3 Effectiveness of our Bounds on Real World Data

We have already discussed conditions under which our bounds for ϵ-`ORBIT` provide a good approximation to `ORBIT`. We now test the effectiveness of our bounds on real-world data. We consider stocks in the S&P 500, and consider four 1-year windows to test the effectiveness of our bounds. We also provide a table that reports the average value of our bounds.

Monte Carlo Simulations, Rayleigh Quotient Interpretation

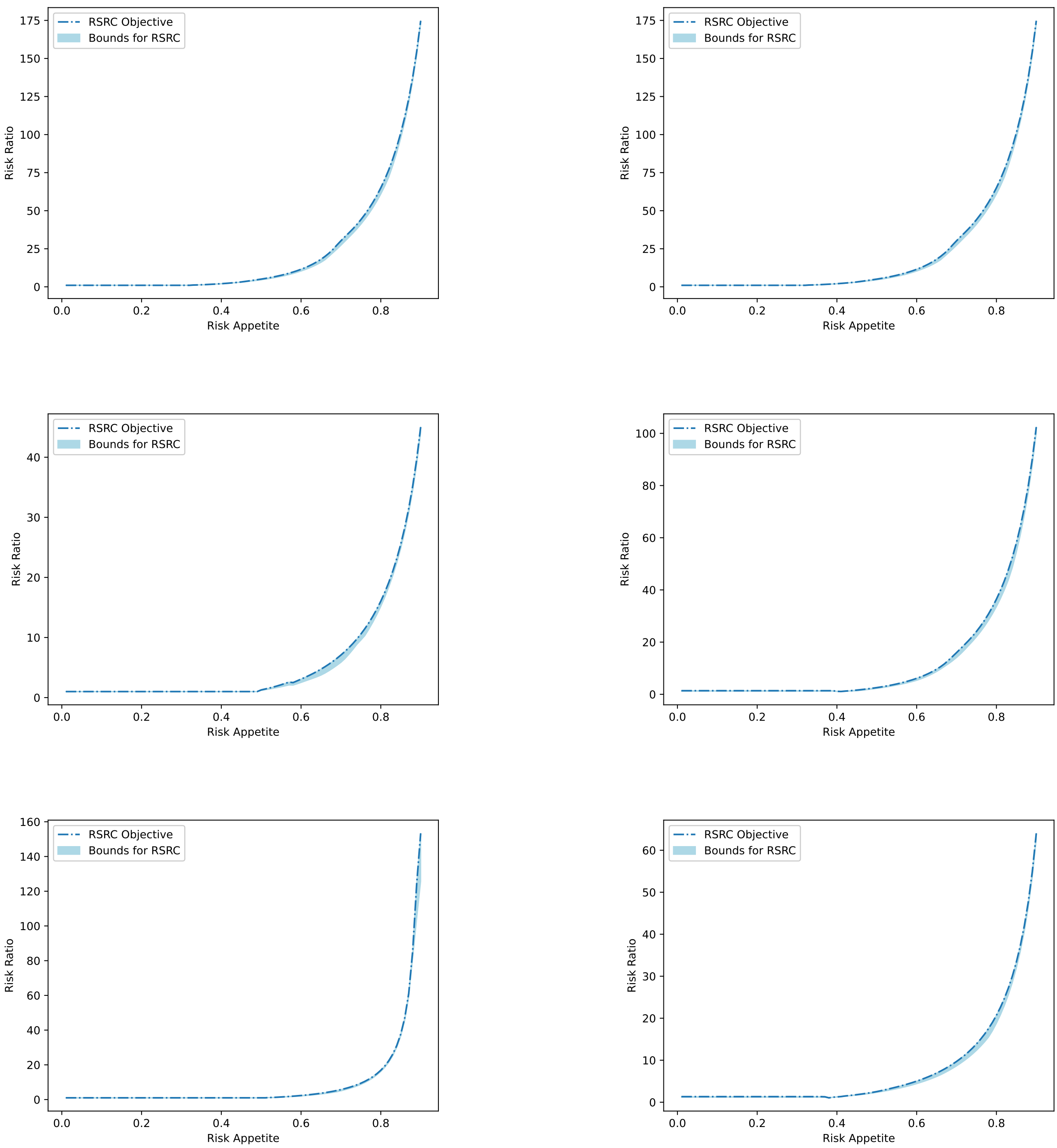

Figure 7: Evolution of $\Theta(x) = \max_{i,j} \frac{\mathcal{RC}_i}{\mathcal{RC}_j}$ over an S&P 500 portfolio along with proven bounds

4.4 Performance on S&P 500 and Mutual Funds

We now demonstrate portfolio performance on real world S&P 500 data of our algorithms. We fix our asset data and sequentially increase our risk appetite across the following algorithms- `LIRA`, ϵ-`ORBIT`,`CVaR-LIRA`, `MVO`, $1/n$.

Table 3: Average value of $\left\{1+\frac{n\max_i\left(\sqrt{\sum_k \Sigma_{k,i}^2}-\Sigma_{i,i}\right)\sum_{i,j}\Sigma^{-1}_{i,j}}{2}\right\}$ and $|\lambda|\left(n\frac{x_p^\top x_p}{x_p^\top \Sigma x_p}-\frac{1}{\lambda^*}\right)$ over 200 cycles

Algorithm	Number of Cycles Implemented	Average value of our bound	Average value of our bound on $\Theta^R(x)-1$ in the risk parity regime
ϵ-`ORBIT`	200	1.031	0.234

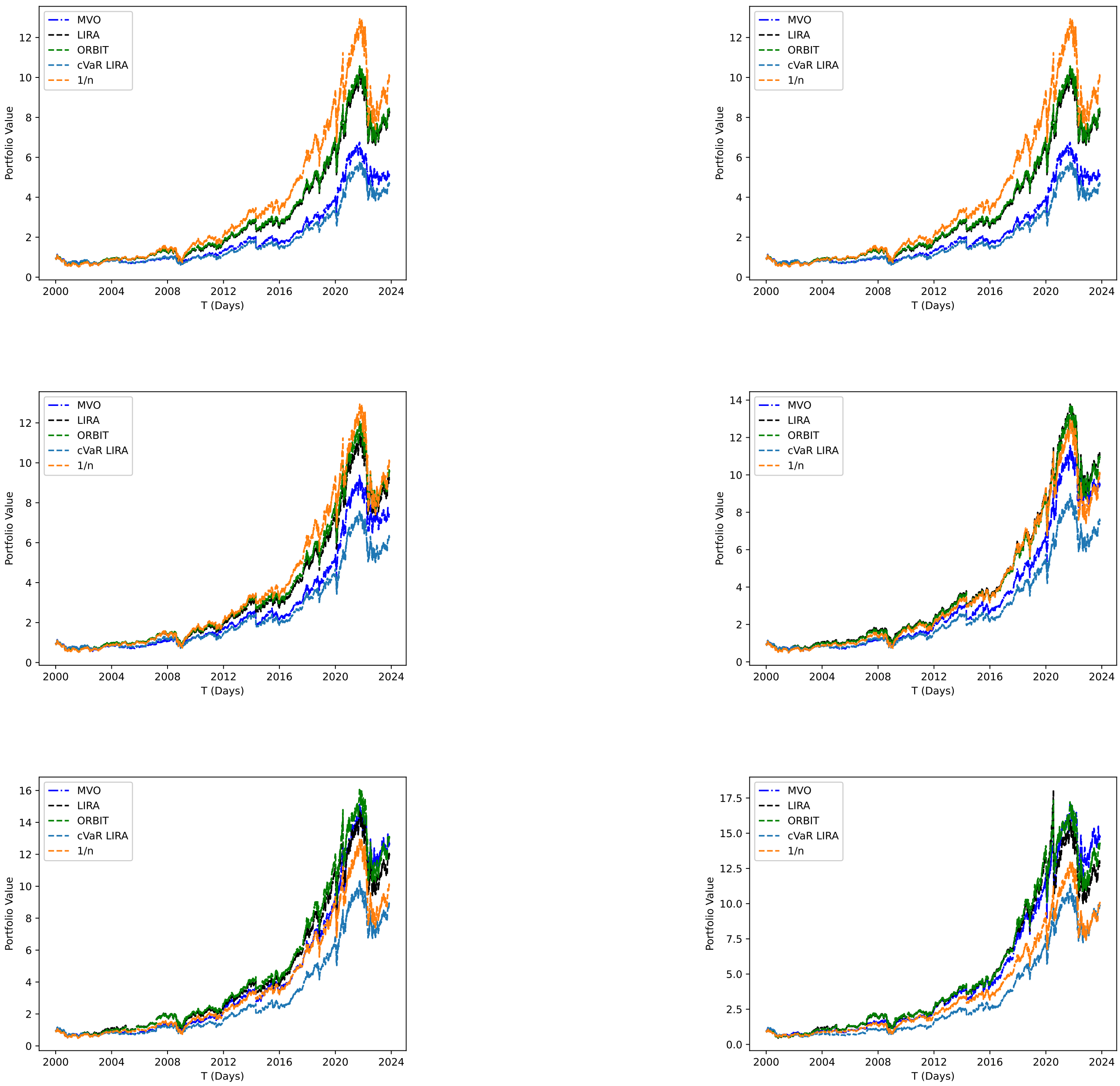

Figure 8: Evolution of portfolios for `LIRA`, ϵ-`ORBIT`,`CVaR-LIRA`, `MVO`, $1/n$ over an S&P 500 for increasing γ. γ increasing left to right, top to bottom

We also provide statistics of the portfolios across different risk appetites,

Table 4: Financial Statistics for γ=0.1, 0.6

Algorithm	Portfolio Value	Sharpe	MDD
ϵ-**ORBIT**	8.121	3.194	0.412
LIRA	8.091	3.193	0.413
MVO	4.013	2.923	0.331
cVaR-**LIRA**	3.912	2.69	**0.312**
$1/N$	**9.912**	**3.211**	0.551

Algorithm	Portfolio Value	Sharpe	MDD
ϵ-**ORBIT**	15.131	**3.356**	0.342
LIRA	12.613	3.273	0.339
MVO	**15.312**	3.263	0.371
cVaR-**LIRA**	10.012	3.08	**0.251**
$1/N$	9.912	3.211	0.551

5 Proofs of Propositions

Theorem 9. *(LIRA) There exists an efficient algorithm $\mathcal{A}$ that finds x^* solving $\max_{x\in\mathcal{P}^R}\min_j \frac{\mathcal{RC}_j(x)}{\sum_{i=1}^n \mathcal{RC}_i(x)}$, for risk measure $R(x)=\sqrt{x^\top \Sigma x}$*

Proof. First we rewrite our optimization problem using expressions for $\mathcal{RC}_j(x)$ (volatility to be our risk measure)

$$\max_{x\in\mathcal{P}^R}\min_j \frac{\mathcal{RC}_j(x)}{\sum_{i=1}^n \mathcal{RC}_i(x)} \iff \max_{x\in\mathcal{P}^R}\min_j \frac{x_j(\Sigma x)_j}{x^\top \Sigma x} \qquad (\mathbf{P})$$

As we will see in the next theorem, our objective is a quadratic form,

$$x_j(\Sigma x)_j = x^\top \Sigma^{(j)} x$$

where,

$$\Sigma^{(i)}_{k,l} = \begin{cases} \frac{\Sigma_{i,l}}{2} & k=i, \quad l\neq i \\ \frac{\Sigma_{i,l}}{2} & l=i, \quad k\neq i \\ \Sigma_{i,i} & i=k=l \\ 0 & \text{o.w.} \end{cases}$$

We show in the next theorem that this matrix has a negative eigenvalue and the quadratic form therefore in general is not convex. Hence we need to convert our problem into a convex one. To this extent, we use the epigraph trick to obtain,

$$\max_{x\in\mathcal{P}^R}\min_j \frac{x_j(\Sigma x)_j}{x^\top\Sigma x} \quad \underbrace{\iff}_{1} \quad \begin{aligned} &\max_{x\in\mathcal{P}^R} \frac{u^2}{x^\top\Sigma x} \\ &\text{s.t. } u^2 \leq x_j(\Sigma x)_j \end{aligned} \qquad (\mathbf{P_1})$$

The first equivalence (1) uses the fact that, in an optimal solution (x^*, u^*) in the right hand side problem,

$$u^{*2} = \min_j x^*_j(\Sigma x^*)_j$$

Now rewriting the constraint in problem $\mathbf{P_1}$ as a norm,

$$\iff \max_{x\in\mathcal{P}^R} \frac{u^2}{x^\top \Sigma x} \quad \underbrace{\iff}_{2} \quad \max_{x\in\mathcal{P}^R} \frac{u^2}{x^\top \Sigma x}$$

$$\text{s.t.} \ \sqrt{\left(u^2 + (\Sigma x)_j^2 + x_j^2\right)} \leq (x_j + (\Sigma x)_j) \qquad \text{s.t.} \ \left\| \begin{pmatrix} u \\ x_j \\ (\Sigma x)_j \end{pmatrix} \right\|_2 \leq x_j + (\Sigma x)_j \tag{$\mathbf{P_2}$}$$

Here, the second equivalence (2) reformulates our inequality as a conic constraint. We finally introduce a change of variables on $\mathbf{P_2}$

$$\begin{aligned} &\iff \max \frac{1}{z^\top \Sigma z} & \underbrace{\iff}_{3} \quad & \min z^\top \Sigma z \\ \text{s.t.} \ &\left\| \begin{pmatrix} 1 \\ z_j \\ (\Sigma z)_j \end{pmatrix} \right\|_2 \leq z_j + (\Sigma z)_j & \text{s.t.} \ &\left\| \begin{pmatrix} 1 \\ z_j \\ (\Sigma z)_j \end{pmatrix} \right\|_2 \leq z_j + (\Sigma z)_j \\ &\mathbf{1}^\top z = w & &\mathbf{1}^\top z = w \\ &r^\top z \geq Rw & &r^\top z \geq Rw \\ &z \geq 0 & &z \geq 0 \end{aligned} \tag{LIRA}$$

The third equivalence (3) introduces a change of variables, $z = x/u$. To see this equivalence in detail, consider a feasible solution (x, u) for our second problem. Now, apply the following transformation:

$$\frac{x}{u} \mapsto \tilde{z}, \quad \frac{1}{u} \mapsto \tilde{w}$$

It is easy to see that

$$\left\| \begin{pmatrix} 1 \\ \tilde{z}_j \\ (\Sigma\tilde{z})_j \end{pmatrix} \right\|_2 \leq \tilde{z}_j + (\Sigma\tilde{z})_j \iff \left\| \begin{pmatrix} u \\ x_j \\ (\Sigma x)_j \end{pmatrix} \right\|_2 \leq x_j + (\Sigma x)_j$$

$$\mathbf{1}^\top \tilde{z} = \tilde{w} \iff \mathbf{1}^\top x = 1$$

$$r^\top \tilde{z} \geq R\tilde{w} \iff r^\top x \geq R$$

Moreover,

$$\frac{1}{\tilde{z}^\top \Sigma \tilde{z}} = \frac{u^2}{x^\top \Sigma x}$$

It is easily seen that, $\tilde{z}, \tilde{w}$ is feasible for the third problem, more so with the same objective value. Conversely, consider (z, w) feasible for the fourth problem, consider transformation,

$$\frac{z}{w} \mapsto \tilde{x}$$
$$\frac{1}{w} \mapsto \tilde{u}$$

Again, it follows very easily that $\tilde{x}, \tilde{u}$ are feasible for the original problem with the same objective cost. Hence these problems are equivalent. The fourth equivalence follows trivially, and our optimization problem is convex. All of this shows that

$$\mathbf{P} \iff \texttt{LIRA}$$

□

Corollary 10. *Let $r_{\texttt{ERC}}$ be return of an* ERC *portfolio, with Σ. If $R \leq r_{\texttt{ERC}}$, and x^* is an optimal solution for* LIRA, *then $x^* = x_p$, where x_p is an* ERC *portfolio given Σ.*

Proof. Trivially,

$$\frac{\mathcal{RC}_j(x)}{\sum_{i=1}^{n} \mathcal{RC}_i(x)} \leq \frac{1}{n}$$

If $R \leq r_{\texttt{ERC}}$, x_p is feasible for `LIRA`. Moreover,

$$\frac{\mathcal{RC}_j(x)}{\sum_{i=1}^{n}\mathcal{RC}_i(x)} = \frac{1}{n}$$

which completes the proof

$\square$

Theorem 11. *let x_* be an optimal solution to ϵ-`ORBIT` with objective value Θ_*, x^* be an optimal solution to `ORBIT` with objective value Θ^*. Moreover, assume that,*

$$x^{*\top}x^* \leq \mathbb{E}\left[x^\top x\right], \quad x \sim Dir(1, \ldots, 1)$$

Then x_* is a $\left\{1 + \left(\sqrt{\sum_k \Sigma_{k,i}^2} - \Sigma_{i,i}\right)\sum_{i,j}\Sigma_{i,j}^{-1}\right\}$ approximation to `ORBIT`, i.e.

$$\Theta(x_*) \leq \alpha(\Sigma, R)\Theta(x^*)$$

where,

$$\alpha(\Sigma, R) = \left\{1 + \left(\sqrt{\sum_k \Sigma_{k,i}^2} - \Sigma_{i,i}\right)\sum_{i,j}\Sigma_{i,j}^{-1}\right\}$$

Proof. We first reformulate our problem using the epigraph trick as in theorem 1, $\min_{x \in = \mathcal{P}^R} \max_{i,j} \frac{\mathcal{RC}_j(x)}{\mathcal{RC}_j(x)}$ as,

$$\begin{aligned} \min_{x\in\mathcal{P}^R} \quad & \frac{t^2}{u^2} \\ \text{s.t.} \quad & t^2 \geq x_j(\Sigma x)_j \\ & u^2 \leq x_j(\Sigma x)_j \end{aligned} \tag{ORBIT}$$

Now notice that, the second constraint can be written as a conic constraint for formulating a convex problem,

$$u^2 \leq x_j(\Sigma x)_j \iff \left\|\begin{pmatrix} u \\ x_j \\ (\Sigma x)_j \end{pmatrix}\right\|_2 \leq x_j + (\Sigma x)_j$$

However, the first constraint poses problems as it can be potentially non-convex (exterior of a convex region). Here we use a simple trick by adding a term to our constraint,

$$t^2 \geq x_j(\Sigma x)_j - \boldsymbol{x}^\top \boldsymbol{\lambda I x}$$

where, $\lambda = \min_i E_-(\Sigma^{(i)})$, $E_-(A)$ computes the smallest eigenvalue of A and Σ_i is the matrix given by,

$$\Sigma_{k,l}^{(i)} = \begin{cases} \frac{\Sigma_{i,l}}{2} & k = i, \quad l \neq i \\ \frac{\Sigma_{i,l}}{2} & l = i, \quad k \neq i \\ \Sigma_{i,i} & i = k = l \\ 0 & \text{o.w.} \end{cases}$$

Here $\Sigma^{(i)}$ is such that, $x_j(\Sigma x)_j = x^\top \Sigma^{(i)} x$, and our extra term , $\lambda x^\top I x$ ensures that our first constraint is convex. This is because the RHS of the constraint is,

$$x^\top \left(\Sigma^{(i)} - \lambda I\right) x$$

Now the jth eigenvalues of $\Sigma^{(i)} - \lambda I$ are,

$$\lambda_j^{(i)} - \lambda = \underbrace{\lambda_j^{(i)} - \min_l E_-(\Sigma^{(l)}) \geq}_{\lambda_j^{(i)} \geq E_-(\Sigma^i) \geq \min_l E_-(\Sigma^{(l)})} 0$$

Making $\Sigma^{(i)} - \lambda I \succeq 0$, and our constraint positive. Now we can formulate an SOCP approximation to our original problem as,

$$\iff \min_{x\in\mathcal{P}^R} \frac{t^2}{u^2} \quad \underbrace{\iff}_{1} \quad \min_{x\in\mathcal{P}^R} \frac{t^2}{u^2}$$

$$\text{s.t.} \quad t^2 \geq x_j(\Sigma x)_j - \lambda x^\top I x \qquad\qquad \text{s.t.} \quad t^2 \geq x^\top \left(\Sigma^{(i)} - \lambda I\right) x$$

$$u^2 \leq x_j(\Sigma x)_j \qquad\qquad u^2 \leq x_j(\Sigma x)_j \tag{$\mathbf{P}$}$$

Here equivalence (1) follows from the definition of $\Sigma^{(i)}$. Now we have a non-convex objective, $\frac{t^2}{u^2}$. However, we can resolve this via dividing the contraints in $\mathbf{P}$ by t,

$$\iff \min_{x\in\mathcal{P}^R} \frac{t^2}{u^2} \quad \underbrace{\iff}_{2} \quad \min \frac{1}{v^2}$$

$$\text{s.t.} \quad 1 \geq \left(\frac{x}{t}\right)^\top \left(\Sigma^{(i)} - \lambda I\right)\left(\frac{x}{t}\right) \qquad\qquad \text{s.t.} \quad 1 \geq z^\top \left(\Sigma^{(i)} - \lambda I\right) z$$

$$\frac{u^2}{t^2} \leq \frac{x_j}{t}(\Sigma \frac{x}{t})_j \qquad\qquad v^2 \leq z_j(\Sigma z)_j$$

$$\gamma^2 w^2 \leq z_j(\Sigma z)_j$$

$$\mathbf{1}^\top z = w$$

$$r^\top z \geq Rw \tag{$\mathbf{P_2}$}$$

In 2, we use a similar idea as in the first theorem, we perform a change of variables to easily notice the forward direction,

$$\frac{x}{t} \mapsto \tilde{z}, \quad \frac{u}{t} \mapsto \tilde{v}, \quad \frac{1}{u} \mapsto \tilde{w}$$

Conversely for the reverse implication we have the following transformation,

$$\frac{v}{w} \mapsto \tilde{u}, \quad \frac{1}{w} \mapsto \tilde{t}, \quad \frac{z}{w} \mapsto \tilde{x}$$

The feasibility is preserved with the same objective value, implying the equivalence. Our objective is now convex, however we can convert it to a second order conic program (SOCP), by introducing variable y in $\mathbf{P_2}$

$$\iff \min \; y \quad \underbrace{\iff}_{3} \quad \min \; y$$

$$\text{s.t.} \quad 1 \geq z^\top \left(\Sigma^{(i)} - \lambda I\right) z \qquad\qquad \text{s.t.} \quad 1 \geq \left\| (\Sigma^{(i)} - \lambda I)^{\frac{1}{2}} z \right\|_2$$

$$\left\| \begin{pmatrix} v \\ z_j \\ (\Sigma z)_j \end{pmatrix} \right\|_2 \leq z_j + (\Sigma z)_j \qquad\qquad \left\| \begin{pmatrix} v \\ z_j \\ (\Sigma z)_j \end{pmatrix} \right\|_2 \leq z_j + (\Sigma z)_j$$

$$\left\| \begin{pmatrix} \gamma w \\ z_j \\ (\Sigma z)_j \end{pmatrix} \right\|_2 \leq z_j + (\Sigma z)_j \qquad\qquad \left\| \begin{pmatrix} \gamma w \\ z_j \\ (\Sigma z)_j \end{pmatrix} \right\|_2 \leq z_j + (\Sigma z)_j$$

$$\frac{1}{v} \leq y \qquad\qquad \left\| \begin{pmatrix} 1 \\ y \\ z \end{pmatrix} \right\|_2 \leq z + v$$

$$r^\top z \geq Rw \qquad\qquad r^\top z \geq Rw$$

$$\mathbf{1}^\top z = w \qquad\qquad \mathbf{1}^\top z = w$$

$$(\epsilon\text{-}\mathtt{ORBIT})$$

Here 3 follows by writing $1/v \leq y$ as a conic constraint. All of this implies,

$$\mathbf{P} \iff \epsilon\text{-}\mathtt{ORBIT}$$

Since we added an extra term $(-\lambda x^\top I x)$ to the optimization problem, we must confirm the interpretability of our problem, i.e. ensure that we actually measure $\max_{i,j} \frac{\mathcal{RC}_i(x)}{\mathcal{RC}_j(x)}$.

First constraint on $\mathbf{P}$ will necessarily pick $\max_i x_i(\Sigma x)_i$ and set $t^2 = \max_i x_i(\Sigma x)_i - \lambda x^\top x$ since $\lambda x^\top x$ is independent of the index, and the second constraint will necessarily pick, $\min_j x_j(\Sigma x)_j$ and set $u^2 = \min_j x_j(\Sigma x)_j$. If $t^2 >$

$\max_i x_i^*(\Sigma x^*)_i - \lambda x^{*T}x^*$, for optimal x^*, we can improve our objective value by decreasing the value of t and hence our inequality must be tight, the same logic follows for u. This implies that our objective value becomes,

$$\frac{x_I(\Sigma x)_I - \lambda x^\top x}{x_J(\Sigma x)_J}$$

where I, J are such that,

$$\begin{aligned}
\frac{\mathcal{RC}_I(x)}{\mathcal{RC}_J(x)} &= \frac{\max_i \mathcal{RC}_i(x)}{\min_j \mathcal{RC}_j(x)}, \quad \forall i, j \\
\implies \frac{\mathcal{RC}_I(x)}{\mathcal{RC}_J(x)} &= \max_{i,j} \frac{\mathcal{RC}_i(x)}{\mathcal{RC}_j(x)}
\end{aligned}$$

Now we prove the approximation bounds on our solution x for ϵ-`ORBIT`, consider a solution x^* for $\min_{x \in = \mathcal{P}^R} \max_{i,j} \frac{\mathcal{RC}_j(x)}{\mathcal{RC}_j(x)}$. Note that $(x, u, t) = (x^*, \sqrt{\min_i x_i^*(\Sigma x^*)_i}, \sqrt{\max_j x_j^*(\Sigma x)_j^*) - \lambda x^{*T}x}$ is a feasible candidate,

$$\begin{aligned}
\mathcal{O}_* &= \frac{(x_*)_I(\Sigma x_*)_I + |\lambda| x_*^\top x_*}{(x_*)_J(\Sigma x_*)_J} \\
&= \Theta_* \left(1 + \frac{|\lambda| x_*^\top x_*}{(x_*)_I(\Sigma x_*)_I}\right) \\
&\leq \mathcal{O}(x^*)
\end{aligned}$$

Here $\mathcal{O}(x)$ is objective value of ϵ-ORBIT for portfolio x and I, J as defined in the proof of theorem 1. Now call, $\Theta = \max_{i,j} \frac{\mathcal{RC}_i(x)}{\mathcal{RC}_j(x)}, \quad \Theta^* = \max_{i,j} \frac{\mathcal{RC}_i(x^*)}{\mathcal{RC}_j(x^*)}$. Now from the optimality of x_*

$$\begin{aligned}
\Theta_* \left(1 + \min_i \min_x \frac{|\lambda| x^\top x}{x_i(\Sigma x)_i}\right) &\leq \Theta_* \left(1 + \frac{|\lambda| x_*^\top x_*}{(x_*)_I(\Sigma x_*)_I}\right) \\
&= \Theta_* \left(1 + \frac{|\lambda| x_*^\top x_*}{x_*^\top \Sigma^{(I)} x_*}\right) \\
&\leq \Theta^* \left(1 + \frac{|\lambda| x^{*T}x^*}{(x^*)_{I^*}(\Sigma x^*)_{I^*}}\right) \\
&\leq \Theta^* \left(1 + \frac{|\lambda|}{(x^*)_{I^*}(\Sigma x^*)_{I^*}}\right), \quad \because x^{*T}x^* \leq 1 \\
&\leq \Theta^* \left\{1 + \frac{|\lambda|}{\left(\frac{x^{*T}\Sigma x^*}{n}\right)}\right\} \\
&\leq \Theta^* \left\{1 + \frac{n|\lambda|}{\min_{x \in \mathcal{P}^R} x^\top \Sigma x}\right\}
\end{aligned}$$

The last inequality follows since,

$$\begin{aligned}
x^\top \Sigma x &= \sum_{i=1}^n x_i(\Sigma x)_i \\
&\leq n \left\{\max_i x_i(\Sigma x)_i)\right\}
\end{aligned}$$

$$\implies \frac{x^\top \Sigma x}{n} \leq \max_i x_i(\Sigma x)_i$$

Now,

$$\min_{y^\top Qy \geq 0} \frac{y^\top y}{y^\top Qy} = \frac{1}{\max_{y^\top Qy \geq 0, \|y\|_2 = 1} y^\top Qy}$$

It can easily be shown that, $\max_{y^\top Qy \geq 0, \|y\|_2 = 1} y^\top Qy$ is the size of the largest `positive` eigenvalue of Q. We first write down the Lagrangian,

$$\mathcal{L}(x, \mu, \nu) = y^\top Qy + \mu(y^\top Qy) - \nu(y^\top y)$$

$$
\begin{aligned}
\nabla \mathcal{L}(x,\mu,\nu) &= 2(1+\mu)Qy - 2\nu y = 0 \\
\implies \qquad Qy &= \frac{\nu}{1+\mu} y, \quad \because \mu \geq 0
\end{aligned}
$$

This implies, $\frac{\nu}{1+\mu}$ must be an eigenvalue, moreover,

$$
y^\top Q y = \frac{\nu}{1+\mu} y^\top y = \frac{\nu}{1+\mu}
$$

Which implies that the objective value is the largest positive eigenvalue of Q. This implies,

$$
\Theta_* \left(1 + \min_i \frac{|\lambda|}{E_+(\Sigma^{(i)})}\right) \leq \Theta^* \left\{1 + \frac{n|\lambda|}{\min_{x\in\mathcal{P}^R} x^\top \Sigma x}\right\}
$$

To finish our proof, we compute the eigenvalues of $\Sigma^{(i)}, \forall i$.

$$
\Sigma^{(i)} = \begin{bmatrix}
0 & \cdots & \frac{\Sigma_{1,i}}{2} & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{\Sigma_{i,1}}{2} & \cdots & \Sigma_{i,i} & \cdots & \frac{\Sigma_{i,n}}{2} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & \frac{\Sigma_{n,i}}{2} & \cdots & 0
\end{bmatrix}
$$

We now perform two operations on $\Sigma^{(i)} - \lambda I$, we fist swap the ith row with the last row and then ith column with the last column (two swaps ensure that the determinant of our new matrix is the same as $\Sigma^{(i)} - \lambda I$), to get M, where,

$$
-M = \begin{bmatrix}
\lambda & 0 & \cdots & 0 & \frac{\Sigma_{i,1}}{2} \\
0 & & \cdots & 0 & \frac{\Sigma_{i,2}}{2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \lambda & \frac{\Sigma_{i,n-1}}{2} \\
\frac{\Sigma_{i,1}}{2} & \frac{\Sigma_{i,2}}{2} & \cdots & \frac{\Sigma_{i,n-1}}{2} & \lambda - \Sigma_{i,i}
\end{bmatrix}
$$

We can write $-M$ as the following block matrix,

$$
-M = \begin{bmatrix}
D & u \\
u^\top & \lambda - \Sigma_{i,i}
\end{bmatrix}
$$

Where $D = \lambda I \in \mathbb{R}^{(n-1)\times(n-1)}$, and $u \in \mathbb{R}^{n-1}$ such that,

$$
u_j = \begin{cases}
\Sigma_{i,j} & j \neq i \\
\Sigma_{i,n} & j = i
\end{cases}
$$

Now we can use Schur's lemma to compute the determinant of $-M$ given by,

$$
\begin{aligned}
\det(-M) &= \det(D)\left(\lambda - \Sigma_{j,j} - u^\top D^{-1} u\right) \\
&= \lambda^{n-1}\left(\lambda - \Sigma_{j,j} - u^\top D^{-1} u\right) \\
&= \lambda^{n-1}\left(\lambda - \Sigma_{j,j} - \frac{1}{\lambda} u^\top u\right) \\
&= \lambda^{n-1}\left(\lambda - \Sigma_{j,j} - \frac{1}{\lambda}\sum_{j\neq i}\frac{\Sigma_{j,i}^2}{4}\right) = 0
\end{aligned}
$$

$$
\implies \lambda = \frac{\left(\Sigma_{i,i} \pm \sqrt{\sum_j \Sigma_{i,j}^2}\right)}{2}, 0
$$

This computation also allows us to get rid of the assumption, $\lambda < 0$, because $\left(\Sigma_{i,i} - \sqrt{\sum_j \Sigma_{i,j}^2}\right) \leq 0 \implies E(\Sigma^{(i)}) < 0, \forall i$.

Hence putting everything together,

$$|\lambda| = \max_i \left\{ \frac{\sqrt{\sum_k \Sigma_{k,i}^2} - \Sigma_{i,i}}{2} \right\}, \quad \lambda^* = \max_i \left\{ \frac{\sqrt{\sum_k \Sigma_{k,i}^2} + \Sigma_{i,i}}{2} \right\}$$

and it is well known that,

$$\min_{\mathbf{1}^\top x = 1} x^\top \Sigma x = \frac{1}{\sum_{i,j} \Sigma_{i,j}^{-1}}$$

$$\implies \Theta_* \leq \left(\frac{1 + \Delta_1}{1 + \Delta_2} \right) \Theta^*$$

Where,

$$\Delta_1 = \frac{n \max_i \left\{ \sqrt{\sum_k \Sigma_{k,i}^2} - \Sigma_{i,i} \right\} \sum_{i,j} \Sigma_{i,j}^{-1}}{2}, \quad \Delta_2 = \frac{\max_i \left\{ \sqrt{\sum_k \Sigma_{k,i}^2} - \Sigma_{i,i} \right\}}{\max_i \left\{ \sqrt{\sum_k \Sigma_{k,i}^2} + \Sigma_{i,i} \right\}}$$

It is easily seen that,

$$\sqrt{\sum_k \Sigma_{k,i}^2} - \Sigma_{i,i} \geq 0$$

$$\implies \max_i \left\{ \sqrt{\sum_k \Sigma_{k,i}^2} - \Sigma_{i,i} \right\} \geq 0$$

This implies that,

$$\Theta_* \leq \underbrace{\left\{ 1 + \frac{n \max_i \left(\sqrt{\sum_k \Sigma_{k,i}^2} - \Sigma_{i,i} \right) \sum_{i,j} \Sigma_{i,j}^{-1}}{2} \right\}}_{\alpha(\Sigma, R)} \Theta^*$$

which completes our proof. □

Corollary 12. *Let x be an optimal solution to above problem* ***P'****, let x^* be an optimal solution* ***P****. Define $\epsilon = 2\lambda \|x - x^*\|_2$, then we have that*

$$\min_{x \in \mathcal{P}^R} \max_i x_i (\Sigma x)_i \leq \min_{x^* \in \mathcal{P}^R} \max_i x_i^* (\Sigma x^*)_i + \epsilon$$

That is, the optimality gap is given by ϵ.

Proof. First we restate problem **P'**

$$\begin{aligned} \min \quad & \gamma \\ \text{s.t.} \quad & \gamma \geq \left\| (\Sigma^{(i)} - \lambda I)^{\frac{1}{2}} x \right\|_2 \\ & r^\top x = R \\ & \mathbf{1}^\top R = 1 \end{aligned} \tag{$\mathbf{P'}$}$$

Then like in ϵ-**ORBIT**, we see that an optimal x to **P'**,

$$\gamma^2 = \left(\max_i x_i (\Sigma x)_i \right) + \lambda x^\top x$$

Then by optimality of x, we have that,

$$
\begin{aligned}
\max_i x_i(\Sigma x)_i &\leq \max_i x_i^*(\Sigma x^*)_i + \lambda\left(x^{*\top}x^* - x^Tx\right) \\
&= \max_i x_i^*(\Sigma x^*)_i + \lambda\left(\|x^*\|_2^2 - \|x\|_2^2\right) \\
&= \max_i x_i^*(\Sigma x^*)_i + \lambda\left(\Big|\|x^*\|_2 - \|x\|_2\Big|\right)\left(\|x\|_2 + \|x^*\|_2\right) \\
&\leq \max_i x_i^*(\Sigma x^*)_i + \lambda\left(\|x - x^*\|_2\right)\left(\|x\|_2 + \|x^*\|_2\right) \\
&\leq \max_i x_i^*(\Sigma x^*)_i + 2\lambda\left(\|x - x^*\|_2\right)
\end{aligned}
$$

where ϵ is the gap in the norm, $2\|x - x^*\|$

$\square$

Discussing revised bounds: Through corollary 6, we claimed that we were able to improve our bounds on α. Our new bound was stated to be,

$$
\begin{aligned}
\alpha &\leq \left\{1 + \left(\frac{\max_i\left(\sqrt{\sum_k \Sigma_{k,i}^2} - \Sigma_{i,i}\right)}{2}\right)\frac{x^{*\top}x^*}{\min_{x\in\mathcal{P}^R}\max_i x_i(\Sigma x)_i}\right\} \\
&\leq \left\{1 + \left(\frac{\max_i\left(\sqrt{\sum_k \Sigma_{k,i}^2} - \Sigma_{i,i}\right)}{2}\right)\frac{x^\top x + \epsilon_1}{\max\left\{\frac{x^\top \Sigma x}{n}, \max_i x_i'(\Sigma x')_i - \epsilon_2\lambda\right\}}\right\}
\end{aligned}
$$

where x^* is optimal solution to `ORBIT`, x is optimal solution to ϵ-`ORBIT` and x' is optimal solution to **P'**. Moreover, we have a similar assumption on x^*, x as on x'', x', i.e. $\|x^* - x\| \leq 2\epsilon_1$. Here, x'' is optimal solution to **P'**. This follows because,

$$
\begin{aligned}
\left|x^{*\top}x^* - x^\top x\right| &\leq \left(\Big|\|x^*\|_2 - \|x\|_2\Big|\right)\left(\|x\|_2 + \|x^*\|_2\right) \\
&\leq 2\left(\|x - x^*\|_2\right) = \epsilon
\end{aligned}
$$

$$
\implies x^{*\top}x^* \leq x^\top x + \epsilon
$$

This bound is potentially better since we don't divide by the volatility. Dividing by volatility causes two problems, it (1) provides a conservative lower bound ($1/n$) to $\max_i \frac{\mathcal{RC}_i}{\sum_{j=1}^n \mathcal{RC}_j}$ and (2) it provides a conservative lower bound to volatility. We combine both of these steps by simply using corollary 6, on problem **P**. So given that ϵ is small, we should be getting better bounds for α. This improvement however does rely on some knowledge of $\|x - x^*\|$ and is not as general as the bound we derived in corollary 6.

Corollary 13. *Let $\mathbb{V}$ be diagonal matrix of volatilities of Σ and define Σ_γ to be*

$$
\Sigma_\gamma = \gamma\mathbb{V} + (1-\gamma)\Sigma
$$

Then we have that,

$$
\lim_{\gamma\to 1} \quad \alpha(\Sigma_\gamma, R) = 1
$$

Moreover, $\exists\gamma'$ such that, $\forall\gamma \geq \gamma'$,

$$
\alpha(\Sigma, R) \leq \mathcal{O}\left(1 + \frac{(1-\gamma)K}{L - (1-\gamma)M}\right)
$$

for K, L, M that depend on Σ. This also characterizes the rate at which $\alpha(\Sigma, R)$ goes to 1.

Proof. Because, $\sqrt{|a|+|b|} \leq \sqrt{|a|} + \sqrt{|b|}$, using this argument iteratively on $\alpha(\Sigma, R)$,

$$\alpha(\Sigma_\gamma, R) \leq \left(1 + \frac{n \max_i \sum_{j\neq i} |(\Sigma_\gamma)_{i,j}| \sum_{i,j} (\Sigma_\gamma^{-1})_{i,j}}{2}\right)$$

Now, since for distinct i, j,

$$(\Sigma_\gamma)_{i,i} = \Sigma_{i,i}, \quad (\Sigma_\gamma)_{i,j} = (1-\gamma)\Sigma_{i,j}$$

$\exists \Sigma_\gamma$ around $\Sigma_0 = \mathbb{V}$ (or equivalently around $\gamma = 1$), such that,

$$\Sigma_{i,i} > (1-\gamma)\sum_{j\neq i} \Sigma_{i,j}, \quad \forall i$$

Now, since Σ diagonally dominant, we have that, ([5]),

$$\left\|\Sigma_\gamma^{-1}\right\|_\infty \leq \frac{1}{\min_i \left(\Sigma_{i,i} - (1-\gamma)\sum_{j\neq i} \left|\Sigma_{i,j}^{-1}\right|\right)}$$

Now by definition,

$$\left\|\Sigma_\gamma^{-1}\right\|_\infty = \max_i \sum_{j=1}^{n} \left|(\Sigma_\gamma)_{i,j}^{-1}\right| \geq \frac{\sum_{i,j} (\Sigma_\gamma)_{i,j}^{-1}}{n}$$

Putting everything together,

$$\begin{aligned}
\lim_{\gamma\to 0} \alpha(\Sigma_\gamma, R) &\leq \lim_{\gamma\to 0} \left(1 + \frac{n \max_i \sum_{j\neq i} |(\Sigma_\gamma)_{i,j}| \sum_{i,j} (\Sigma_\gamma^{-1})_{i,j}}{2}\right) \\
&\leq \lim_{\gamma\to 0} \left(1 + \frac{n^2 \max_i \sum_{j\neq i} |(\Sigma_\gamma)_{i,j}| \left\|\Sigma_\gamma^{-1}\right\|_\infty}{2}\right) \\
&\leq \lim_{\gamma\to 0} \left\{1 + \frac{n^2 \max_i \sum_{j\neq i} |(\Sigma_\gamma)_{i,j}|}{\min_i \left((\Sigma_\gamma)_{i,j} - \sum_{j\neq i} |(\Sigma_\gamma)_{i,j}|\right)}\right\} = 0
\end{aligned}$$

This also characterizes the rate at which our bound goes to 1, as $\gamma \to 1$

□

Corollary 14. *(Proximity to risk parity) If $R \leq r_{\mathit{ERC}}$, x^* be an optimal solution to ϵ-ORBIT and x_p be a risk parity solution and γ chosen such that x_p feasible for ϵ-ORBIT, then*

$$\max_{i,j} \frac{x_i^*(\Sigma x)_i}{x_j^*(\Sigma x^*)_j} - 1 \leq n\lambda \frac{x_p^\top x_p}{x_p^\top \Sigma x_p}$$

Proof. By assumption and optimality of x^*,

$$\max_{i,j} \frac{x_i^*(\Sigma x)_i}{x_j^*(\Sigma x^*)_j} + |\lambda| \frac{x^{*T} x}{\min_i (x_i^* \Sigma x_i^*)} \leq 1 + n|\lambda| \frac{x_p^\top x_p}{x_p^\top \Sigma x_p}$$

Now trivially,

$$\max_{i,j} \frac{x_i^*(\Sigma x)_i}{x_j^*(\Sigma x^*)_j} \geq 1$$

$$\implies |\lambda| \frac{x^{*T}x}{\min_i(x_i^* \Sigma x_i^*)} \leq n|\lambda| \frac{x_p^\top x_p}{x_p^\top \Sigma x_p}$$

We now have our result,

$$\begin{aligned}
\max_{i,j} \frac{x_i^*(\Sigma x)_i}{x_j^*(\Sigma x^*)_j} - 1 &\leq n|\lambda| \frac{x_p^\top x_p}{x_p^\top \Sigma x_p} - |\lambda| \frac{x^{*T}x}{\min_i(x_i^* \Sigma x_i^*)} \\
&\leq n|\lambda| \frac{x_p^\top x_p}{x_p^\top \Sigma x_p} - |\lambda| \frac{x^{*T}x}{x^{*T}\Sigma x^*} \qquad \because \min_i x_i(\Sigma x)_i \leq \sum_{j=1}^n x_j(\Sigma x)_j = x^\top \Sigma x \\
&\leq n|\lambda| \frac{x_p^\top x_p}{x_p^\top \Sigma x_p} - |\lambda| \min_x \frac{x^T x}{x^T \Sigma x} \\
&\leq n|\lambda| \frac{x_p^\top x_p}{x_p^\top \Sigma x_p} - \frac{|\lambda|}{\lambda^*} \quad \text{(Optimal Rayleigh quotient value)} \\
&\leq |\lambda| \left(n \frac{x_p^\top x_p}{x_p^\top \Sigma x_p} - \frac{1}{\lambda^*} \right)
\end{aligned}$$

□

Theorem 15. *Assume that,*

$$\Theta^L(x_L) \geq \frac{(1-\alpha)\max_j r_j \sum_{i,j} \Sigma_{i,j}^{-1}}{\phi(\Phi^{-1}(\alpha))}$$

and that,

$$\max_j r_j \left(\sum_{i,j} \Sigma_{i,j}^{-1} \right) \leq w_\alpha$$

where Θ^L and x_L are spread metric and solution to `LIRA`*. Moreover, assume $r \geq 0$. Then there exists an efficient algorithm that finds x_*,*

$$\Theta^{CVaR}(x_*) \geq \epsilon \Theta^{CVaR}(x^*)$$

$$\epsilon = \left(\frac{w_\alpha \sqrt{x_*^\top \Sigma x_*}}{CVaR(x_*)} \right) \left(1 - \frac{SR^*}{w_\alpha} \right) \Theta^{CVar}(x^*)$$

Where SR^* is the optimal value of the Sharpe ratio problem,*

$$\max_{x \in \mathcal{P}^R} \frac{r^\top x}{\sqrt{x^\top \Sigma x}}$$

Proof. First we write down `CVaR-LIRA`,

$$\max_{x \in \mathcal{P}^{\mathcal{R},\gamma}} \min_i \mathcal{RC}_i^{\text{CVaR}} = \max_{x \in \mathcal{P}^{\mathcal{R},\gamma}} \Theta^{\text{CVar}}(x) \tag{CVaR-LIRA}$$

We now consider the problem,

$$\begin{aligned}
\max_{x \in \mathcal{P}^{\mathcal{R}}} \quad & \frac{u^2}{w_\alpha x^\top \Sigma x} \\
\text{s.t.} \quad & u_i \leq \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}(\Sigma x)_i - r_i\sqrt{x^\top \Sigma x}, \quad \forall i \\
& u^2 \leq x_i u_i, \quad \forall i
\end{aligned} \tag{P}$$

It is easily seen that the objective value of our problem for optimal x is given by

$$u^2 = \min_i x_i u_i = \frac{\mathrm{CVar}(x)}{w_\alpha \sqrt{x^\top \Sigma x}} \frac{\min_i \left\{ \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha} \frac{x_i (\Sigma x)_i}{\sqrt{x^\top \Sigma x}} - r_i x_i \right\}}{\mathrm{CVar}(x)} = \frac{\mathrm{CVar}(x)}{w_\alpha \sqrt{x^\top \Sigma x}} \Theta^{\mathrm{CVar}}(x)$$

Now, assume $r_i \geq 0$. It is important to ensure that there exists an $x \in \mathcal{P}^R$ such that we have a positive objective value. This is because of our constraint $u^2 \leq x_i u_i$ that else becomes infeasible. From our assumptions,

$$\begin{aligned}
\exists \quad x, \quad \text{s.t.} \quad \Theta^L(x) = \min_j \frac{\mathcal{RC}_j}{\sum_{i=1}^n \mathcal{RC}_i} &\geq \frac{(1-\alpha)\max_l r_l \sum_{i,j} \Sigma^{-1}_{i,j}}{\phi(\Phi^{-1}(\alpha))} \\
\implies \frac{\mathcal{RC}_j}{\sum_{i=1}^n \mathcal{RC}_i} &\geq \frac{(1-\alpha)\max_l r_l \sum_{i,j} \Sigma^{-1}_{i,j}}{\phi(\Phi^{-1}(\alpha))}, \quad \forall j \\
\implies \frac{\mathcal{RC}_j}{\sum_{i=1}^n \mathcal{RC}_i} &\geq \frac{(1-\alpha) r_j \sum_{i,j} \Sigma^{-1}_{i,j}}{\phi(\Phi^{-1}(\alpha))}, \quad \forall j \\
\implies \frac{\mathcal{RC}_j}{\sum_{i=1}^n \mathcal{RC}_i} &\geq \frac{(1-\alpha) r_j}{\phi(\Phi^{-1}(\alpha))\sqrt{x^\top \Sigma x}}, \quad \forall j \\
\implies \frac{\mathcal{RC}_j}{\sum_{i=1}^n \mathcal{RC}_i} &\geq \frac{(1-\alpha) r_j x_j}{\phi(\Phi^{-1}(\alpha))\sqrt{x^\top \Sigma x}}, \quad \forall j \\
\implies \underbrace{\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha} \frac{(\Sigma x)_j}{\sqrt{x^\top \Sigma x}} - r_j}_{u_j} \geq 0, \quad \forall j
\end{aligned}$$

$$\implies \min_i x_i u_i \geq 0$$

Then this problem is equivalent to,

$$\begin{aligned}
\min_z \quad & w_\alpha z^\top \Sigma z \\
\text{s.t.} \quad & u_i \leq \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha} (\Sigma z)_i - r_i \left\| \Sigma^{\frac{1}{2}} z \right\|_2, \quad \forall i \\
& \left\| \begin{pmatrix} u \\ z_j \\ (\Sigma z)_j \end{pmatrix} \right\|_2 \leq z_j + (\Sigma z)_j \\
& r^\top z \geq R w \\
& e^\top z = w
\end{aligned} \tag{\textbf{P}}$$

Here we used change of variable,

$$\frac{x}{u} \to z, \quad \frac{1}{u} \to w$$

Now consider x^*, an optimal solution to `CVar-LIRA`. By optimality of x, we have,

$$\frac{\mathrm{CVar}(x)}{w_\alpha \sqrt{x^\top \Sigma x}} \Theta^{\mathrm{CVar}}(x) \geq \frac{\mathrm{CVar}(x^*)}{w_\alpha \sqrt{x^{*\top} \Sigma x^*}} \Theta^{\mathrm{CVar}}(x^*)$$

This implies that,

$$\Theta^{\mathrm{CVar}}(x) \geq \left(\frac{w_\alpha \sqrt{x_*^\top \Sigma x_*}}{\mathrm{CVaR}(x_*)} \right) \underbrace{\left(\frac{\mathrm{CVar}(x^*)}{w_\alpha \sqrt{x^{*\top} \Sigma x^*}} \right)}_{\gamma} \Theta^{\mathrm{CVar}}(x^*)$$

We now come up with a lower bound for γ,

$$
\begin{aligned}
\gamma &= \frac{\mathrm{CVar}(x^*)}{w_\alpha \sqrt{x^{*\top}\Sigma x^*}} \\
&= 1 - \frac{r^\top x^*}{w_\alpha \sqrt{x^{*\top}\Sigma x^*}} \\
&= 1 - \frac{\mathrm{SR}(x)}{w_\alpha} \\
&\geq 1 - \frac{\mathrm{SR}^*}{w_\alpha}
\end{aligned}
$$

Here, SR^* is the optimal Sharpe ratio for r, Σ. Putting all of this together we have,

$$
\Theta^{\mathrm{CVar}}(x) \geq \left(\frac{w_\alpha \sqrt{x_*^\top \Sigma x_*}}{\mathrm{CVaR}(x_*)} \right) \left(1 - \frac{\mathrm{SR}^*}{w_\alpha} \right) \Theta^{\mathrm{CVar}}(x^*)
$$

□

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